On Dynamical Properties of Quasicrystals

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Abstract

In the first half of this thesis we study the properties of the dynamical hull associated with model sets arising from irregular Euclidean Cut and Project Schemes. We provide deterministic as well as probabilistic constructions of irregular windows whose associated Cut and Project Schemes yield Delone dynamical systems with positive topological entropy. Moreover, we provide a construction of an irregular window whose associated dynamical hull has zero topological entropy but admits a unique ergodic measure.

Furthermore, we show that dynamical hulls of irregular model sets always admit an infinite independence set. Hence, the dynamics cannot be tame. We extend this proof to a more general setting and show that tame implies regular for almost automorphic group actions on compact spaces.

In the second half of this thesis, we provide and discuss a generalization of the Cut and Project formalism. We show that this new formalism yields all sets generated by Euclidean Cut and Project Schemes as well as non-Meyer sets. Furthermore, we give a sufficient criterion to obtain Meyer sets by this formalism in Euclidean space.

Zusammenfassung

Im ersten Teil dieser Arbeit studieren wir die Eigenschaften der dynamischen Hülle von Model Sets, welche durch irreguläre euklidische Cut-and-Project Schemes erzeugt werden. Wir konstruieren (sowohl deterministisch als auch nicht-deterministisch) Beispielklassen irregulärer Fenster, deren zugehörige dynamische Hüllen positive topologische Entropie haben. Ebenso konstruieren wir irreguläre Fenster, deren zugehörige dynamische Hüllen keine topologische Entropie besitzen, jedoch eindeutig ergodisch sind.

Wir zeigen, dass dynamische Hüllen irregulärer Model Sets stets eine unendliche unabhängige Menge beinhalten, wodurch die Dynamik dieser Hüllen nicht zahm sein kann. Im Anschluss benutzen wir die hierzu entwickelten Methoden um zu zeigen, dass zahme fast-automorphe Gruppenaktionen auf kompakten Räumen regulär sein müssen.

Im zweiten Teil der Arbeit entwickeln wir eine Verallgemeinerung des Cut-and-Project Formalismus. Wir zeigen, dass diese Verallgemeinerung sowohl alle Delone Mengen, welche von euklidischen Cut-and-Project Schemes stammen, als auch Delone Mengen, welche nicht die Meyer-Eigenschaft erfüllen, erzeugen kann. Weiterhin zeigen wir ein hinreichendes Kriterium auf, um mit Hilfe dieses neuen Formalismus Meyer-Mengen im euklidischen Raum zu erzeugen.

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Contents

Chapter 1

Introduction

The term *crystal* is usually associated with solid matter whose atoms are arranged in an exactly repeating pattern. Hence, by translational symmetry, knowledge of only a section of this structure provides already knowledge of the whole structure. It is well-known that such crystals only admit diffraction spectra consisting of sharp bright spots, a property which is sometimes referred to as *long-range order*. For a long time, scientific consensus was that also the converse implication should hold, that is, long-range order implies translational symmetry.

However, this changed drastically in 1982, when 2011 Nobel laureate Dan Shechtman discovered solids which admit long-range order without translational symmetry ([SBGC84]). Such solids became known as *quasicrystals* and discussions about precise definitions of terms like "crystal", "long-range order" or even "order" itself arose ([Sen96]).

On the mathematical side, non-periodic geometric structures have been studied at least since the 1960's. Probably most famous are aperiodic tilings of the plane like the *Wang tiling* or the *Penrose tiling*. Interpreting the vertices of such tilings as atoms provides a well-established possibility to describe quasicrystals. This leads to the notion of *Delone sets* [1].

Roughly speaking, such sets are countable point sets such that two points are not too close to each other and such that the space between points is not too large. More precisely, given a locally compact abelian group G (which we equip with a metric d_G for better understanding), we say a subset $\Lambda \subseteq G$ is a Delone set if

- Λ is *uniformly discrete*, that is, there exists some $r > 0$ such that $d_G(x, y) \geq r$ for all $x \neq y \in \Lambda$,
- Λ is *relatively dense*, that is, there exists some $R > 0$ such that $\Lambda \cap B_R(g) \neq \emptyset$ for all $g \in G$.

A common approach to generate Delone sets is the *Cut and Project method* which was introduced by Meyer in 1972 ([Mey72]). We say a triple $(G, H, \mathcal{L})$ consisting of a locally compact abelian and σ-compact group G, a locally compact abelian group H and a lattice $\mathcal{L} \subseteq G \times H$ is a *Cut and Project Scheme (CPS)* if the canonical projection $\pi_G : G \times H \to G$ is one-to-one on $\mathcal{L}$ while for the canonical projection $\pi_H : G \times H \to H$ the image $\pi_H(\mathcal{L})$ is dense in H. Given a compact subset $W \subseteq H$ (the associated *window*), the CPS generates a uniformly discrete subset of G given by

$$\curlywedge(W) = \pi_G \left(\mathcal{L} \cap (G \times W)\right).$$

Such a set is referred to as *model set* and, in case of $\operatorname{int}(W) = \emptyset$, as a *weak model set*. We may refer to G as *physical space* and H as *internal space*, respectively. Figure 1 illustrates the Cut and Project method in the *planar* case $G = H = \mathbb{R}$.

[1] Note that, in the following discussion, we will use terminology which is well-known in literature about aperiodic order. While we provide brief definitions of certain key concepts, we refer the reader unfamiliar with these notations to Part I (in particular Chapter 4) for precise definitions and further background.

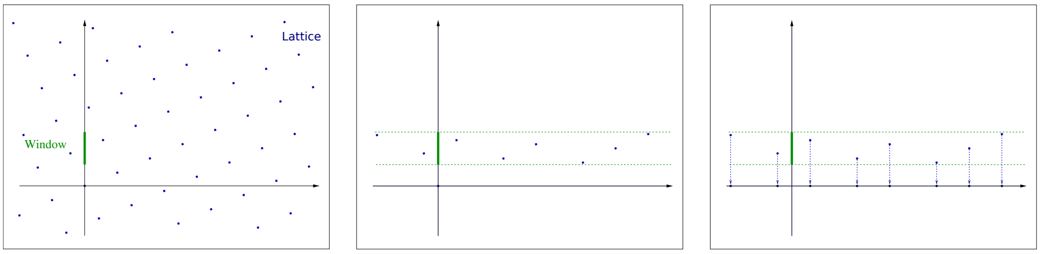

Figure 1: Cut and Project method in case $G = H = \mathbb{R}$

An advantage of this formalism lies in the circumstance that the structure of the window determines many properties of the corresponding model set. For one thing, it is possible to control geometric properties of the model set. The following implications are well-known ([Rob07], [Sch99]):

(i) If the window is *proper*, that is, $\mathrm{cl}(\mathrm{int}(W)) = W$, then $\curlywedge(W)$ is a Delone set.

(ii) If the window is *generic*, that is, $\partial W \cap L^* = \emptyset$, then $\curlywedge(W)$ is repetitive.

(iii) If the window is *regular*, that is, $\Theta_H(\partial W) = 0$, then $\curlywedge(W)$ has uniform patch frequencies. Here, Θ_H denotes the Haar measure on H.

Apart from that, the structure of W also controls the properties of a certain dynamical system associated to $\curlywedge(W)$. To that end, one defines a metric on the space of uniformly discrete sets, where two such sets are close to each other if they - possibly after a small translation - agree on a large compactum. The topology induced by this metric is called *local topology*. This provides the possibility to associate a dynamical system to $\curlywedge(W)$ by considering the G-action

$$(g, \Lambda) \mapsto \Lambda - g$$

on the *dynamical hull* $\Omega(\curlywedge(W)) = \mathrm{cl}\{\curlywedge(W) - g \mid g \in G\}$, where the closure is taken with respect to the local topology. It is well-known that the following implications hold ([Sch99]):

(i) If W is proper, then $\Omega(\curlywedge(W))$ is compact.

(ii) If W is generic, then $(\Omega(\curlywedge(W)), G)$ is minimal.

(iii) If W is regular, then $(\Omega(\curlywedge(W)), G)$ is uniquely ergodic.

Moreover, in case of proper windows, the dynamical hull admits a maximal equicontinuous factor $\mathbb{T} = (G \times H)/\mathcal{L}$. In case of regularity the dynamical hull is even isomorphic to the minimal flow on this factor $\mathbb{T}$ defined by

$$\omega : G \times \mathbb{T} \to \mathbb{T} : (g, \xi) \mapsto \xi + [g, 0]_{\mathbb{T}},$$

and therefore has pure point dynamical spectrum ([Sch99]) as well as zero topological entropy ([BLR07]). Dynamical hulls arising from such *regular model sets* have been intensively studied over the course of the last years. These studies focused also on proving pure point diffraction for regular model sets ([Hof95], [Sch99]) and developing several approaches for that purpose ([LMS02],[BL04]).

Conversely to the observation of regular model sets only admitting hulls with zero entropy, Moody asked whether model sets corresponding to *irregular windows* (that is, $\Theta_H(\partial W) > 0$) admit dynamical hulls with positive topological entropy (compare [PH13] for a historical survey about this question) and Schlottmann suggested in [Sch99] that such hulls cannot be uniquely ergodic anymore.

For instance, in [BMP00] and [HR15] the authors discuss a dynamical system arising from the example of visible lattice points. As it turns out, this system has positive topological entropy and is not uniquely ergodic. Concerning this result it is even more surprising, that, in general, both questions bear negative answers. In [BJL16] the authors constructed an irregular CPS whose corresponding dynamical hull has zero entropy and is uniquely ergodic. Moreover, it was shown that each Toeplitz sequence can be interpreted as a model set - and there are known examples of uniquely ergodic irregular Toeplitz flows with zero topological entropy ([MP79], [Wil84]).

However, both the positive entropy example and zero entropy example rely on the fact of the internal space having a more complex structure then being Euclidean. In fact, the example of visible lattice points uses a p-adic internal space, while the internal space in case of Toeplitz systems is chosen as an odometer. Thus, the questions by Moody and Schlottmann are still open in case of Euclidean Cut and Project Schemes.

In Part II we will focus on these questions. First, in Chapter 5, we are going to construct irregular windows for Euclidean Cut and Project Schemes such that the corresponding dynamical hull admits positive topological entropy. To that end, we will introduce the concept of *embedded fullshifts*, that is, a subset $S \subseteq \mathbb{R}^N$ of positive asymptotic density and a uniformly discrete subset $U \subseteq \mathbb{R}^N$ such that for any subset $S' \subseteq S$ there exists some $\Gamma \in \Omega(\curlywedge(W))$ with $\Gamma \subseteq U$ and $S' = \Gamma \cap S$. This means that we may think of the elements of S as points which can independently of each other be switched on or off without leaving the hull. As it turns out, the existence of such an embedded fullshift is a sufficient condition for $(\Omega(\curlywedge(W)), G)$ to admit positive topological entropy. We will also see that the existence of an embedded fullshift together with a few minor additional assumptions on the window guarantees us to be not uniquely ergodic.

Further, we will show that embedded fullshifts are closely related to the local structure of ∂W around the points of L^*. This will enable us to provide a sufficient condition for positive entropy in terms of the window.

Equipped with these tools, in the remaining chapter we aim to construct irregular windows which yield dynamical hulls with positive topological entropy. A first class of such windows will be obtained by constructing irregular windows in a probabilistic setting. This leads to the following theorem:

Theorem 1 (Theorem 5.18). *Suppose $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ is a Euclidean Cut and Project Scheme and $C \subseteq [0,1]$ is a Cantor set of positive Lebesgue measure. Let $(G_n)_{n\in\mathbb{N}}$ be a numbering of the connected components of $[0,1]\setminus C$ and put $\Sigma^+ = \{0,1\}^{\mathbb{N}}$. Denote by $\mathbb{P}$ the Bernoulli distribution on Σ^+ with equal probability $1/2$ for each symbol and define*

$$W(\omega) = C \cup \bigcup_{n\in\mathbb{N}:\omega_n=1} G_n,$$

where $\omega \in \Sigma^+$. Then for $\mathbb{P}$-almost every $\omega \in \Sigma^+$ the set $W(\omega)$ is proper and the dynamical system $\big(\Omega(\curlywedge(W(\omega)+h)), \mathbb{R}^N\big)$ has positive topological entropy for all $h \in \mathbb{R}$ and is minimal for h from a residual subset of $\mathbb{R}$ (depending on ω).

A second class of irregular windows yielding hulls with positive entropy will be obtained by using methods of rotational dynamics. This approach is justified by the following fundamental observation: each lattice of a Euclidean Cut and Project Schemes $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ is generated by a rotation of $\mathbb{Z}^{N+1}$ via a regular matrix satisfying certain properties. To ensure denseness of L^* we may assume that the $(N+1)$-th row of A consists of rationally independent entries. By assuming that the last entry of this row equals one, we may describe L^* as the lift of N irrational rotations on the circle $\mathbb{R}/\mathbb{Z}$. Now, since L^* arises from a dynamical system, we may exploit this additional structure to construct windows which yield embedded fullshifts.

In contrast to the former construction this approach is fully deterministic.

Theorem 2 (Theorem 5.23). *Suppose $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ is a Euclidean Cut and Project Scheme. Then there exists a proper irregular window $W \subseteq [0,1]$ such that $\big(\Omega(\curlywedge(W+h)), \mathbb{R}^N\big)$ has positive topological entropy for all $h \in \mathbb{R}$.*

Concluding this chapter, by modifying the previous construction slightly to acquire an irregular window with empty interior, we obtain a similar statement for dynamical hulls arising from weak model sets.

Theorem 3 (Theorem 5.25). *Suppose* $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ *is a Euclidean Cut and Project Scheme. Then there exists a compact irregular window* $W \subseteq [0,1]$ *such that* $\big(\Omega(\curlywedge(W+h)), \mathbb{R}^N\big)$ *has positive topological entropy.*

In Chapter 6 we will reconsider the questions by Moody and Schlottmann in the Euclidean setting. As it turns out, we may provide negative answers to both questions. To that end, we introduce two general criteria for an irregular window W to admit dynamical hulls with zero entropy. First, we discuss a self similarity condition for the window, that is, there are just finitely many different local structures around points in $\partial W \cap L^*$. This implies that the fibres of the factor map $\Omega(\curlywedge(W)) \to \mathbb{T}$ are finite, which leads to zero entropy. However, in this case we may have several ergodic measures.

Thus, we also introduce a slightly more technical condition on W such that the associated system still has zero entropy but allows infinite fibres. In this case, we are even uniquely ergodic.

The remaining chapter is dedicated to the construction of irregular windows for planar CPS satisfying the conditions above. Using methods of rotational dynamics we obtain an irregular Cantor set in $[0,1]$, whose gaps can be filled in two ways - one way generates a window satisfying the self similarity condition, while the other way provides a window which yields a uniquely ergodic dynamical hull with zero topological entropy.

As a consequence of the discussions in Sections 6.1 and 6.3 we obtain

Theorem 4. *Suppose* $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ *is a Euclidean Cut and Project Scheme. Then there exists a proper irregular window* $W \subseteq [0,1]$ *such that* $(\Omega(\curlywedge(W)), \mathbb{R})$ *has zero topological entropy.*

In the remaining part of the chapter we will discuss how to apply the methods above to higher dimensional CPS $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ with irregular window $W \subseteq \mathbb{R}$. It turns out that the structure of the dynamical hull arising from the higher dimensional CPS is reflected in the structure of the dynamical hulls arising from certain planar CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L}_i)$, $i = 1, \dots, N$, which have the same window W. Here, the two dimensional lattices $\mathcal{L}_i$ arise from the $N+1$-dimensional lattice $\mathcal{L}$. We obtain

Theorem 5 (Proposition 6.25). *Suppose* $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ *is a Euclidean Cut and Project Scheme with irregular window* $W \subseteq \mathbb{R}$. *If there exists an associated planar Cut and Project Scheme* $(\mathbb{R}, \mathbb{R}, \mathcal{L}_i)$ *with window* $W \subseteq \mathbb{R}$ *such that the entropy arising from the hull associated to this planar CPS is zero, then the entropy of* $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ *vanishes.*

In the final Chapter 7 of Part II we will focus on other dynamical properties of dynamical hulls rather then entropy. To that end, we consider a CPS $(G, H, \mathcal{L})$ consisting of second-countable locally compact abelian groups G and H. We say $\Omega(\curlywedge(W))$ admits an *infinite free set* $S \subseteq G$ if there exists a uniformly discrete set $\Lambda \subseteq G$ such that $S \subseteq G$ and for all subsets $S' \subseteq S$ there exists some $\Gamma \in \Omega(\curlywedge(W))$ such that $\Gamma \cap S = S'$. Observe that embedded fullshifts discussed earlier are a special case of infinite free sets.

As it turns out, the existence of an infinite free set implies the existence of an *independence pair* in $(\Omega(\curlywedge(W)), G)$, that is, a concept introduced in topological dynamics equivalent to *non-tameness* of the system ([KL07]).

Theorem 6 (Theorem 7.7). *Let* $(G, H, \mathcal{L})$ *be a Cut and Project Scheme where* G *as well as* H *are second-countable. If* $W \subseteq H$ *is a proper irregular window, then* $(\Omega(\curlywedge(W)), G)$ *admits an infinite free set and is therefore not tame.*

Hence, in the context of Cut and Project Schemes, the question of tameness of a dynamical system translates into a topological question regarding the structure of the window. With minor modifications it is possible to use the methods established for the proof of the previous theorem to show a similar statement concerning almost automorphic dynamical systems.

Theorem 7 (Theorem 7.11). *Let X be a compact topological space G be an arbitrary topological group. Suppose (X,G) is almost automorphic. If (X,G) is tame, then it is a regular extension of its maximal equicontinuous factor.*

This provides a positive answer to a question asked by Glasner ([Gla18, Problem 5.7]).

After this short excursion to the general theory of topological dynamics we return to Cut and Project Schemes.

It is well-known that all model sets are quite restrictive in the sense of possessing many additional structures: model sets are always FLC sets and model sets always satisfy the *Meyer property* ([Rob07], [HR15]), that is, there exists a finite set $F \subseteq G$ such that

$$\curlywedge(W) - \curlywedge(W) \subseteq F + \curlywedge(W).$$

However, in recent years interest grew in studying properties of point sets not satisfying the Meyer property or FLC ([FR14], [HR15], [Fra15], [LS18]). In both cases, such point sets cannot arise from Cut and Project Schemes.

In Part III we aim to generalize the well-known Cut and Project method to a new formalism which allows more flexibility in the construction of point sets. To that end, we consider a minimal dynamical system (X,T), where $T = (T,\cdot)$ is a discrete abelian subgroup of some other group T'. Further, we fix a window $W \subseteq X$ and a starting point $x_0 \in X$.

We consider all elements of $t \in T$ such that $t \cdot x_0$ hits the window. In case of a $\mathbb{Z}$-action one could think about obtaining bi-infinite sequences in $\{0,1\}^{\mathbb{Z}}$. However, instead of working with sequences, we want to map the points of $\{t \in T \mid t \cdot x_0 \in W\}$ into some *physical space*, which usually is a locally compact abelian group $G = (G,+)$. To that end, we need some function which maps the times in some "nice" way to G, i.e., the images of this function have to be somehow compatible with the given dynamical system. It turns out that such maps are provided by *cocycles*, that is, a function $\varphi : T \times X \to G$ such that the equation

$$\varphi_{st}(x) = \varphi_s(x) + \varphi_t(s \cdot x)$$

holds for all $s,t \in T$ and $x \in X$. We refer to the tuple (X,T,G,φ) as a *dynamical Cut and Project Scheme* and associate to it a *dynamical model set*

$$\curlywedge_{x_0}^{T}(W) = \{\varphi_t(x_0) \mid t \cdot x_0 \in W\}.$$

The main objective of Part III is to introduce this method and discuss its basic properties. We also aim to give examples of point sets which may arise from dynamical Cut and Project Schemes but not from classical Cut and Project Schemes.

In Chapter 8, we will give the necessary background of cocycles and precise definitions of the new concepts above. In particular, we will focus on the properties of the cocycle which heavily influences the structure and properties of the obtained point set.

The first two sections of Chapter 9 are devoted to provide sufficient conditions for dynamical Cut and Project Schemes to yield repetitive dynamical model sets or dynamical model sets with uniform patch frequencies, respectively. Those properties do not only depend on the windows boundary as in the classical case but also on the choice of the cocycle. Note that we will also give a criterion for almost repetitivity in case of non-FLC sets.

The third Section 9.3 has its focus on dynamical Cut and Project Schemes with Euclidean physical space. In this setting we will see that dynamical model sets do not necessarily possess the Meyer property. Here, we will focus on cocycles which are induced by N step functions of the form

$$f : X \to \mathbb{R}^N : x \mapsto \sum_{j=1}^{K} a_j \cdot \chi_{X_j}(x),$$

where $a_j \in \mathbb{R}^N$ and $\{X_1, \dots, X_K\}$ is a partition of X. By choosing the coefficients carefully the cocycle generated by such a function provides Delone sets with FLC. However, in contrast to classical Cut and Project Schemes, the Meyer property depends on more factors. In this case, it is the structure of the underlying dynamical system.

To that end, we introduce additional terminology. We say a subset $B \subseteq X$ of a measure space (X, μ) is a *bounded remainder set (with respect to x_0)* if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $x \in \mathcal{O}_{\mathbb{Z}}(x_0)$ we have

$$\left| \sum_{i=0}^{n-1} \chi_B(H^i(x)) - n\mu(B) \right| \leq C,$$

as well as

$$\left| \sum_{i=n}^{-1} \chi_B(H^i(x)) - |n|\mu(B) \right| \leq C$$

for all $n \in -\mathbb{N}$. Further, we say a partition $\mathcal{X}$ of X *consists of bounded remainder sets* if each element of $\mathcal{X}$ is a bounded remainder set. The concept of bounded remainder sets is for instance discussed in [GL15] and can be dated back to [Ost27]. Recently, such kinds of sets also appeared in the context of classical Cut and Projects Schemes ([FG18], [HK16], [HKK17], see also [FHK18]).

This concept enables us to provide a sufficient condition for dynamical model sets arising under certain cocycles to be Meyer sets in the setting of rotations induced by $\mathbb{Z}^N$ on a compact space X.

Theorem 8 (Corollary 9.16)**.** *Consider the dynamical Cut and Project Scheme $(X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ with proper window $W \subseteq X$ and starting point $x_0 \in X$. Assume that $(X, \mathbb{Z}^N)$ carries a unique ergodic measure. Let the cocycle φ be generated by N step functions defined on partitions $\mathcal{X}^i$, $i = 1, \ldots, N$, of X. If each partition $\mathcal{X}^i$ consists of bounded remainder sets, then $\curlywedge_{x_0}^{\mathbb{Z}^N}(W)$ is a Meyer set.*

In Chapter 10 we will discuss whether dynamical hulls arising from dynamical model sets have a factor system. While for classical Cut and Project Schemes the factor is given by the Kronecker flow on the "torus" $\mathbb{T} = (G \times H)/\mathcal{L}$ (and thus depends on the lattice $\mathcal{L}$), in the generalized setting the factor space heavily depends on the cocycle. Given a dynamical Cut and Project Scheme (X, T, G, φ), the cocycle induces certain mappings $h_t : G \times X \to G \times X$ which are crucial to define the *suspension of X (with respect to φ)*

$$S_\varphi(X) = (G \times X)/\{h_t \mid t \in T\}.$$

In this construction, we will slightly generalize ideas used for cocycles $\varphi : \mathbb{Z}^M \times X \to \mathbb{R}^N$ first appearing in [FKMS93] and [KMMS98].

In our setting it turns out that the suspension is a compact G-space. Further, for a factor map to exist, we will need the window W to satisfy a certain regularity condition (*irredundancy*). It turns out that this condition is kind of a dynamical equivalent to irredundancy in the case of classical Cut and Project Schemes.

Theorem 9 (Proposition 10.14)**.** *Let (X, T, G, φ) be a dynamical Cut and Project Scheme with proper irredundant window $W \subseteq X$ and starting point $x_0 \in X$. Then for each Λ such that $\curlywedge_{x_0}^{T}(\mathrm{int}(W)) \subseteq \Lambda \subseteq \curlywedge_{x_0}^{T}(W)$ there exists a semiconjugation*

$$\beta : (\Omega(\Lambda), G) \to (S_\varphi(X), G).$$

The last Chapter 11 is dedicated to show that this new formalism is indeed compatible with classical Cut and Project Schemes in case of Euclidean spaces. We aim to proof that all model sets acquired from Euclidean Cut and Project Schemes can also be obtained from dynamical Cut and Project Schemes.

Theorem 10 (Theorem 11.5)**.** *Given an Euclidean Cut and Project Scheme $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$, there exists a dynamical Cut and Project Scheme $(X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ such that for any window $W \subseteq \mathbb{R}^M$ there exists a window $V \subseteq X$ such that we have*

$$\curlywedge(W) = \curlywedge_0^{\mathbb{Z}^N}(V).$$

We would like to mention that the results of Part II have been published in [JLO16] and [FGJO18].

Notation and Standing Assumptions

Unless mentioned otherwise, we will use the following notation throughout this thesis:

- The cardinality of a set A is denoted by $\sharp A$.
- The symmetric difference of two sets A, B is denoted by $A \Delta B$.
- The M-dimensional standard torus is denoted by $\mathbb{T}^M$ and the 1-dimensional standard torus (i.e., the circle) by $\mathbb{S} = \mathbb{T}^1$.
- The Lebesgue measure on $\mathbb{R}^N$ as well as on $\mathbb{T}^N$ will be denoted by Leb.
- The Haar measure on a group G is denoted by Θ_G. Sometimes we will also use the notations $\sharp(.) = \Theta_{\mathbb{Z}^N}(.)$ as well as $|.| = \mathrm{Leb}(.)$.
- The space of regular $N \times N$-matrices over $\mathbb{R}$ will be denoted by $\mathrm{GL}(N, \mathbb{R})$.
- Depending on the context, $X \cong Y$ means that the spaces X and Y are isomorphic or homeomorphic, respectively.
- The indicator function of a set A is denoted by $\chi_A(.)$.
- The metric on a space X will be denoted by d_X.
- Given any vector space X, the convex hull of a subset $A \subseteq X$ is denoted by $\mathrm{Conv}(A)$.
- An arbitrary norm in Euclidean space is denoted by $\|.\|$.
- The space of continuous functions $f : X \to Y$ will be denoted by $C(X, Y)$. If Y is known from the context we will just write $C(X)$.
- For functions $f, g : X \to Y$ between metric spaces, we write $f = o(g)$ if for each $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that we have $d_Y(0, f(x)) \leq \varepsilon \cdot d_Y(0, g(x))$ for all $x \in X$ with $d_X(0, x) > N$.

Unless mentioned otherwise, we will stick to the following assumptions throughout this thesis:

- Topological spaces are supposed to be compact and Hausdorff.
- Topological groups are supposed to be locally compact second-countable abelian groups.
- For discrete subgroups or lattices acting on a topological space, we will use multiplicative notation for both the operation on the group and the action on the space. In the context of (dynamical) Cut and Project Schemes, physical spaces will usually be endowed with an additive notation.

Part I

Preliminaries

Chapter 2

Group Theory

In this chapter we want to provide some advanced concepts of topological group theory. Our main goal is to establish the notion of *averaging sequences* in locally compact abelian groups. This concept is crucial for ergodic theory in groups, which will be discussed in detail in Section 3.7.

2.1 Topological Groups and Lattices

In this short section we want to recall some basic notations and properties concerning topological groups.

Let $G = (G, +)$ be a locally compact abelian (lca) group. We say G is *compactly generated* if there exists a relatively compact neighbourhood $U \subseteq G$ of the identity such that $G = \bigcup_{n \in \mathbb{N}} U^n$, where $U^n = \left\{ \sum_{j=1}^{n} u_j : u_j \in U \right\}$. In this case we call U *generating neighbourhood*.

Theorem 2.1 (Structure Theorem for compactly generated lca groups, [HR12])**.** *Let G be a compactly generated lca group. Then $G \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$, where $n, m \in \mathbb{N}$ and K is some compact group.*

The Theorem of Birkhoff-Kakutani (compare for instance [BK96]) yields that lca groups are metrizable if they are first-countable and Hausdorff. In case G is even second-countable the metric on G has some sophisticated properties.

Lemma 2.2 ([CdlH16])**.** *Let G be a locally compact and second-countable group. Then*

(i) G is σ-compact.

(ii) There exists a (left-)G-invariant and proper metric d_G on G which induces the topology on G.

(iii) (G, d_G) is a complete metric space.

Remark 2.3. A metric d_G being *proper* means that for all $\varepsilon > 0$ and $g \in G$ the balls

$$B_\varepsilon(g) = \{h \in G \mid d_G(g, h) < \varepsilon\}$$

are relatively compact.

By $\widehat{G} = \mathrm{Hom}(G, \mathbb{S})$ we denote the set of all continuous group homomorphisms from G to $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. We call $\lambda \in \widehat{G}$ a *character* of G and refer to $\widehat{G}$ as *dual group* of G. Indeed, $\widehat{G}$ carries a group structure and it is possible to construct a topology on $\widehat{G}$.

Proposition 2.4 ([RS00])**.** *Let G be a locally compact group. Then $\widehat{G}$ is a topological group itself.*

In case G is even a compact group, the following well-known theorem holds.

Theorem 2.5 (Peter-Weyl Theorem, [Sim96]). *Let G be a compact abelian group. Then the characters of G form an orthonormal basis of $L^2(G, \Theta_G)$.*

Let G be a lca group and suppose $T \leq G$ is a *discrete subgroup* of G, that is, there exists a neighbourhood $U \subseteq G$ of the identity 0 of G such that $T \cap U = \{0\}$. It is easy to see that the following holds.

Lemma 2.6. *Let G be a lca group and T a discrete subgroup.*

(i) T is closed in G.

(ii) T is locally compact.

(iii) Every compact subset of T is finite.

Now suppose T is a discrete subgroup of G. We call T a *lattice* in G if the quotient space G/T is compact. In this case, there exists a unique Borel measure μ on G/T (up to scaling) which satisfies

(i) $\mu(G/T) < \infty$,

(ii) for all $g \in G$ and for all measurable $U \subseteq G/T$ we have $\mu(g + U) = \mu(U)$.

Note that lattices inherit properties of the group they are contained in. In particular we have

Lemma 2.7 ([CdlH16]). *Let T be a lattice in a topological group G. If G is locally compact and second-countable, then T is also locally compact and second-countable.*

Let T be a lattice in G. Suppose $F \subseteq G$ is a Borel set such that every $g \in G$ can be written uniquely as $g = f + t$ for some $f \in F$ and $t \in T$. Then we call F *fundamental domain* of T in G. Moreover, we have

Lemma 2.8 ([KK98]). *Suppose T is a lattice in a locally compact abelian group G. Then there exists a relatively compact fundamental domain of T in G.*

As an important example we consider the case $G = \mathbb{R}^N$. Then, given N linearly independent vectors $v_1, \ldots, v_N \in \mathbb{R}^N$, the set

$$T = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_N$$

is a lattice in $\mathbb{R}^N$. We will refer to $\{v_1, \ldots, v_N\}$ as a *basis* for T and to the vectors $v_i \in \mathbb{R}^N$ as *primitive vectors* of T. Observe that the matrix $(v_1, \ldots, v_N) \in \mathrm{GL}(N, \mathbb{R})$ provides an isomorphism between $\mathbb{Z}^N$ and T. On the other hand, each lattice $T \leq \mathbb{R}^N$ defines a regular matrix given by its base vectors. In particular, we obtain that all lattices in Euclidean space are isomorphic to $\mathbb{Z}^N$. Furthermore, a given lattice T in $\mathbb{R}^N$ might be described by different sets of primitive vectors, hence there is no uniquely determined set of primitive vectors associated with T.

A natural choice for a fundamental domain of T is

$$\left\{ \sum_{i=1}^{N} t_i v_i : t_i \in [0, 1) \text{ for all } i \in \{1, \ldots, N\} \right\}.$$

It is well-known that the volume of this fundamental domain is given by $|\det(v_1, \ldots, v_N)|$ and equals $\mu(\mathbb{R}^N/T)$, where μ is the measure on $\mathbb{R}^N/T$ induced by the Lebesgue measure Leb on $\mathbb{R}^N$. Although the choice of a fundamental domain is not unique, it is not hard to see that all fundamental domains of a given lattice have the same volume.

2.2 Averaging Sequences

Suppose $G = (G, +)$ is a locally compact and σ-compact group. Fix a left Haar measure Θ_G on G. Suppose $(S_n)_{n\in\mathbb{N}}$ is a sequence of compact and non-empty subsets of G. We say $(S_n)_{n\in\mathbb{N}}$ is *increasing* if $S_{n-1} \subseteq S_n$ holds for all $n \in \mathbb{N}$. We call $(S_n)_{n\in\mathbb{N}}$ *exhausting* if $G = \bigcup_{n\in\mathbb{N}} S_n$.

We say an increasing and exhausting sequence $(F_n)_{n\in\mathbb{N}}$ of compact, non-empty subsets sets of G is a *Følner sequence* if

$$\lim_{n\to\infty} \frac{\Theta_G(F_n \Delta (g + F_n))}{\Theta_G(F_n)} = 0$$

for all $g \in G$.

A related but more restrictive concept is that of van Hove sequences. For compact $A, K \subseteq G$ we call

$$\partial^K A = ((K + A) \setminus \mathrm{int}(A)) \cup ((K + \mathrm{cl}(A^c)) \setminus A^c)$$

the K*-boundary* of A. An increasing and exhausting sequence $(A_n)_{n\in\mathbb{N}}$ of compact, non-empty subsets of G is called *van Hove sequence* if

$$\lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \Theta_G(\partial^K A_n) = 0$$

for every compact $K \subseteq G$.

We say an increasing and exhausting sequence $(D_n)_{n\in\mathbb{N}}$ of compact, non-empty subsets of G is *tempered* (or satisfies *Shulman's condition*) if there exists some $C \geq 1$ such that

$$\Theta_G\left(\bigcup_{k=1}^{n-1} D_n - D_k\right) \leq C\Theta_G(D_n)$$

for all $n \in \mathbb{N}$.

We refer to both Følner sequences and van Hove sequences as *averaging sequences*. A direct consequence of these definitions is

Lemma 2.9. *Let G be a locally compact group. Then every van Hove sequence is also a Følner sequence.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a van Hove sequence in G and fix $g \in G$. Since $(g + A_n)^c \subseteq g + A_n^c$ we obtain

$$A_n \setminus (g + A_n) \subseteq (g + A_n^c) \setminus A_n^c \subseteq (g + \mathrm{cl}(A_n^c)) \setminus A_n^c.$$

Further, we have

$$(g + A_n) \setminus A_n \subseteq (g + A_n) \setminus (\mathrm{int}(A_n)).$$

This implies $A_n \Delta (g + A_n) \subseteq \partial^{\{g\}} A_n$. Hence, (A_n) is Følner. □

Suppose $L \subseteq G$ is a compact set and we "thicken" a Følner sequence $(F_n)_{n\in\mathbb{N}}$ by L, i.e. we consider a sequence $(L + F_n)_{n\in\mathbb{N}}$. Then these "thickened" versions still have asymptotically the same volume as the original sequence. A similar statement holds for "thickened" versions of the K-boundary of van Hove sequences.

Lemma 2.10 ([MR13])**.** *Let G be a locally compact group and suppose $L \subseteq G$ is compact.*

(i) For any Følner sequence $(F_n)_{n\in\mathbb{N}}$ in G we have

$$\Theta_G(L + F_n) = \Theta_G(F_n) + o(\Theta_G(F_n)) \text{ as } n \to \infty.$$

(ii) For any van Hove sequence $(A_n)_{n\in\mathbb{N}}$ in G and any compact $K \subseteq G$ we have

$$\Theta_G(L + \partial^K A_n) = o(\Theta_G(A_n)) \text{ as } n \to \infty.$$

In addition, the van Hove property is preserved by additions with compact sets, i.e.,

Lemma 2.11 ([Str05])**.** *Let G be a locally compact group and $(A_n)_{n\in\mathbb{N}}$ a van Hove sequence in G. Suppose $K \subseteq G$ is compact. Then*

(i) $B_n = \mathrm{cl}\left(A_n \setminus \partial^K A_n\right)$

(ii) $C_n = A_n + (K \cup \{0\})$

are van Hove sequences in G. Moreover, we have

$$\lim_{n\to\infty} \frac{\Theta_G(B_n)}{\Theta_G(A_n)} = \lim_{n\to\infty} \frac{\Theta_G(C_n)}{\Theta_G(A_n)} = 1.$$

The question arises whether groups admit averaging or tempered averaging sequences. In the case of lca σ-compact groups the answer is positive for both cases.

Lemma 2.12 ([Sch99])**.** *Every σ-compact locally compact abelian group admits a van Hove sequence (and thus a Følner sequence).*

Lemma 2.13 ([Lin99], [MR13])**.** *Let G be a σ-compact locally compact abelian group. Then every Følner sequence admits a tempered subsequence.*

Remark 2.14. In fact, the existence of a Følner sequence in a group G implies that G is *amenable*, that is, the group is carrying a mean function which is invariant under translation by group elements. Although the concept of amenability plays an important role in many branches of mathematics, it will be not important for our further discussions. We refer to [Ocn06] or [Zim13] for a detailed overview on this topic.

Chapter 3

Theory of Dynamical Systems

This chapter will provide solid background in dynamical systems as well as ergodic theory. Both concepts will play a crucial role in the later parts of this thesis.

3.1 Topological Dynamical Systems

Let X be a compact topological space and $G = (G, +)$ be a topological group with identity 0. A *(left) G-action on X* is a function $f : G \times X \to X$ such that

(G1) $f(0, x) = x$ for all $x \in X$,

(G2) $f(g, f(h, x)) = f(g + h, x)$ for all $x \in X$ and $g, h \in G$.

We say a pair (X, G) is a *(topological) dynamical system* if X is a compact Hausdorff space, G acts on X and

$$f : G \times X \to X : x \mapsto f(g, x)$$

is continuous. Put $f_g : X \to X : x \mapsto f(g, x)$ for $g \in G$.

For $g, h \in G$, it is immediate that $f_g \circ f_h = f_{g+h}$ and $f_g \circ f_{-g} = f_e$. Hence, each f_g is a homeomorphism of X onto itself with $(f_g)^{-1} = f_{-g}$. Throughout this thesis, we will mostly use the notation $g \cdot x = f_g(x) = f(g, x)$.

Remark 3.1. Of course, it is possible to define *right G-actions on X* in the obvious way, i.e., there is a function $f : X \times G \to X$ such that

(G1') $f(x, 0) = x$ for all $x \in X$,

(G2') $f(f(x, h), g) = f(h + g, x)$ for all $x \in X$ and $g, h \in G$.

In case $f_g : X \to X : x \mapsto f(x, g)$ is continuous, one may use the notation $x \cdot g = f_g(x) = f(x, g)$. However, throughout this thesis all occurring non-abelian group actions will be left actions. In case G is abelian we will not have to distinguish between left and right group actions and just speak about *group actions*.

We refer to the pair $(X, \mathbb{Z}^n)$ as a *discrete dynamical system* and to the pair $(X, \mathbb{R}^n)$ as a *flow*. In the special case $G = \mathbb{Z}$, we might represent the action via some homeomorphism

$$T : X \to X : x \mapsto T(x)$$

in the sense of $n \cdot x = T^n(x)$, where $T^n = \underbrace{T \circ \ldots \circ T}_{n \text{ times}}$. In this case, we might refer to T as *transformation*.

We say a dynamical system (X, T) is *free*, if $g \cdot x = x$ implies $g = 0$. Throughout this thesis we will assume that all occurring dynamical systems are free (unless mentioned otherwise).

We say a subset $S \subseteq X$ is *G-invariant* if $g \cdot S = S$ for all $g \in G$. A topological dynamical system is called *minimal* if the only closed, non-empty and invariant subset of X is X itself. Equivalently, X is minimal if for all $x \in X$ the *orbit* $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$ is dense in X.

A set $A \subseteq G$ is called *syndetic* if there exists a compact subset $K \subseteq G$ such that

$$G = A + K = \{a + k \mid a \in A, k \in K\}.$$

In case of $G \in \{\mathbb{Z}, \mathbb{R}\}$ this might be interpreted as A not containing arbitrarily large gaps. We say that a point $x \in X$ is an *almost periodic point* if the *set of return times*

$$N(x, U) = \{g \in G \mid g \cdot x \in U\}$$

is syndetic for every open neighbourhood U of x. Then we have

Lemma 3.2 ([Aus88])**.** *Let (X, G) be a minimal topological dynamical system. Then every point $x \in X$ is almost periodic.*

Theorem 3.3 (Gottschalk's Theorem, [Pet83])**.** *Let (X, G) be a topological dynamical system and $x_0 \in X$ such that $\mathcal{O}(x_0)$ is dense in X. Then (X, G) is minimal if and only if $N(x_0, U)$ is syndetic for all open neighbourhoods U of x_0.*

Suppose (X, G) and (Y, G) are topological dynamical systems. A map $\beta : X \to Y$ is called a *G-map* if $\beta(g \cdot x) = g \cdot \beta(x)$ for all $g \in G$, $x \in X$. We say (Y, G) is a *factor* of (X, G) (and (X, G) is an *extension* of (Y, G)) if there exists a continuous and surjective G-map $\beta : X \to Y$. Such a map is called *semiconjugation* or *factor map*. In case β is a bijective semiconjugation, we call β *conjugation* or *isomorphy*.

3.2 Measurable Group Actions

Let X be a compact topological space and $G = (G, +)$ a topological group. Let $\mathcal{B} = \mathcal{B}(X)$ denote the Borel σ-algebra over X. A *measurable (left) G-action on X* is defined by a $(\mathcal{B}(G) \times \mathcal{B}(X))$ - $\mathcal{B}(X)$-measurable map $f : G \times X \to X$ such that f satisfies (G1) and (G2).

Remark 3.4. The discussions in Remark 3.1 also apply to the definition of measurable group actions.

We say a probability measure $\mu : \mathcal{B} \to [0, 1]$ is *invariant (with respect to G)* if

$$\mu(g \cdot A) = \mu(A)$$

for all $A \in \mathcal{B}$ and $g \in G$. In this case, (X, G, μ) is called a *measure-preserving dynamical system*. We call an invariant measure μ *ergodic (with respect to G)* if

$$\mu(A) \in \{0, 1\}$$

for all G-invariant $A \in \mathcal{B}$. For fixed ergodic measure μ, we refer to both the G-action and the dynamical system (X, G, μ) as *ergodic*. In case there exists exactly one ergodic probability measure μ on X, we say the system (X, G, μ) is *uniquely ergodic* and μ is an *uniquely ergodic measure*.

We say two measure-preserving dynamical systems (X, G, μ) and (Y, G, ν) are *measure-theoretically isomorphic* if there exist full measure sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a measurable bijection $\eta : X_0 \to Y_0$ such that

$$\eta(g \cdot x) = g \cdot \eta(x)$$

for all $x \in X_0$ and $g \in G$.

3.3 Equicontinuous Dynamical Systems

In this section we discuss a special class of topological dynamical systems. Assume that (X, d) is a compact metric space and (X, G) a dynamical system. We say the metric d is *G-invariant* if for all $g \in G$, $x, y \in X$ holds

$$d(g \cdot x, g \cdot y) = d(x, y).$$

A dynamical system (X, G) which admits a G-invariant metric is called *equicontinuous*. In case G is abelian, equicontinuous systems yield some additional structure.

Lemma 3.5 ([Aus88]). *Let (X, G) be a minimal equicontinuous dynamical system and assume G is abelian. Then X can be given the structure of a topological group $(X, \odot)$ with a G-invariant metric. Moreover, G can be naturally embedded as a dense subgroup of X, that is, there exists a group homomorphism $\phi : G \to X$ such that $\phi(G)$ is a dense subgroup of X and $\phi(g) \odot x = g \cdot x$ for all $g \in G$.*

Remark 3.6. (i) Sometimes we will refer to such systems as described in Lemma 3.5 as *group rotations*.

(ii) We want to point out that $(X, \odot)$ is indeed an abelian group. This follows directly by Theorem 3.12 stated in the next section. In the same theorem there will also be a statement about the structure of non-abelian minimal equicontinuous dynamical systems.

Lemma 3.7 ([Bro76]). *Let (X, G) be a minimal equicontinuous dynamical system. Then this system is uniquely ergodic.*

As an easy example we consider an *irrational rotation*, that is, the compact metric space X given by the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ equipped with a $\mathbb{Z}$-action defined via an homeomorphism

$$H : \mathbb{S} \to \mathbb{S} : x \mapsto x + \alpha \mod \mathbb{S},$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. It is well-known that $(\mathbb{S}, \mathbb{Z})$ is a minimal equicontinuous dynamical system ([KH97]). By the above Lemma, $(\mathbb{S}, \mathbb{Z})$ carries a unique ergodic measure which is given by the Lebesgue measure on $\mathbb{S}$ ([KH97]).

A dynamical system (Y, G) is called *maximal equicontinuous factor (MEF)* of (X, G) if it is an equicontinuous factor of (X, G) and has the property that every other equicontinuous factor (Z, G) of (X, G) is also a factor of (Y, G).

It turns out that each dynamical system has a MEF.

Lemma 3.8 ([Aus88]). *Suppose X is a compact metric space and (X, G) a dynamical system. Then (X, G) has a unique (up to conjugacy) a maximal equicontinuous factor (Y, G) with Y compact metric.*

A continuous map $\beta : X \to Y$ is called *almost one-to-one* if the set of its *injectivity points*

$$X_0 = \{x \in X \mid \beta^{-1}(\{\beta(x)\}) = \{x\}\}$$

is dense in X. In case β is a factor map between topological dynamical systems (X, G) and (Y, G) we refer to β as *almost one-to-one factor map* and to (X, G) as *almost one-to-one extension* of (Y, G).

Corollary 3.9 ([Dow05]). *Let (X, G) and (Y, G) be topological dynamical systems and let $\beta : (X, G) \to (Y, G)$ be an almost one-to-one factor map. Then the following holds.*

(i) Both X_0 and $\beta(X_0)$ are invariant subsets of X and Y, respectively.

(ii) Both X_0 and $\beta(X_0)$ are residual in X and Y, respectively.

(iii) If (Y, G) is minimal, then (X, G) is minimal.

Now suppose (Y, G) is a minimal equicontinuous dynamical system. By Lemma 3.5 and Lemma 3.7 the system yields a unique ergodic measure which equals the Haar measure Θ_Y. We say an almost one-to-one extension (X, G) of (Y, G) is *regular* if $\Theta_Y(\beta(X_0)) > 0$, otherwise we call it *irregular*. The following facts are easy to see.

Corollary 3.10. *Let (X, G) be a regular almost one-to-one extension of a minimal equicontinuous dynamical system (Y, G) with factor map β. Then the following holds.*

(i) We have $\Theta_Y(\beta(X_0)) = 1$.

(ii) The system (X, G) is uniquely ergodic.

(iii) The systems (X, G) and (Y, G) are measure-theoretically isomorphic.

Proof. (i) follows directly from Corollary 3.9(i) and ergodicity of Θ_Y.

(ii). Assume μ and ν are distinct ergodic measures on X. Then μ and ν are mutually singular. Note that both measures project to Θ_Y. Together with (i) this yields that X_0 has full measure with respect to μ and ν, respectively, which contradicts mutually singularity of μ and ν. Hence, X is uniquely ergodic.

(iii). This follows directly from (i) and (ii). □

We call a topological dynamical system (X, G) *almost automorphic* if it is an almost one-to-one extension of its MEF and the MEF is minimal. Note that, by our considerations above, the latter implies that (X, G) is minimal, too.

3.4 The Ellis Semigroup and Tame Systems

Given a topological dynamical system (X, G), the *Ellis semigroup* associated to (X, G) is defined as

$$\mathcal{E}(X) = \mathrm{cl}\left(\{x \mapsto g \cdot x \mid g \in G\}\right) \subseteq X^X,$$

where the closure is taken with respect to the product topology. The operation on $\mathcal{E}(X)$ is given by composition of maps.

Lemma 3.11 ([Aus88])**.** *Let (X, G) be a topological dynamical system. Then $(\mathcal{E}(X), G)$ is a topological dynamical system, where G acts via $\tau \mapsto g \cdot \tau$.*

Regarding minimal and equicontinuous dynamical systems we obtain even more.

Theorem 3.12 ([Aus88])**.** *Suppose (X, G) is a minimal equicontinuous dynamical system. Then the following holds.*

(i) $\mathcal{E}(X)$ is a compact metrizable abelian topological group. Further, (X, G) and $(\mathcal{E}(X), G)$ are conjugated.

(ii) Assume that G is not abelian. Then (X, G) is a factor of $(\mathcal{E}(X), G)$, where the factor map is given by

$$\pi : \mathcal{E}(X) \to X : \tau \mapsto \tau \cdot x$$

for some fixed $x \in X$. In particular, π is open.

Remark 3.13. Suppose (X, G) is a minimal equicontinuous dynamical system. By Lemma 3.7, this system yields a unique ergodic measure μ. In case G is abelian, we may assume $X = \mathcal{E}(X)$ and hence obtain $\mu = \Theta_{\mathcal{E}(X)}$. In case G is not abelian, we obtain $\mu = \Theta_{\mathcal{E}(X)} \circ \pi^{-1}$, where π is as in Theorem 3.12(ii) and $\Theta_{\mathcal{E}(X)}$ denotes the left Haar measure on $\mathcal{E}(X)$. Note that $\mathcal{E}(X)$ is compact and thus unimodular.

A given topological dynamical system (X, G) is called *tame* if

$$\sharp\mathcal{E}(X) \leq 2^{\aleph_0},$$

and *non-tame* or *wild* otherwise. Here, $2^{\aleph_0}$ denotes the cardinality of the continuum.

Theorem 3.14 ([Gla18]). *Suppose (X, G) is a minimal and tame topological dynamical system which allows for an invariant measure. Then (X, G) is an almost one-to-one extension of its maximal equicontinuous factor.*

For later purposes the following characterization of tame systems will be useful. Let U_0 and U_1 be closed subsets of X such that $U_0 \cap U_1 = \emptyset$. We call the pair (U_0, U_1) an *independence pair* if there exists an infinite set $S \subseteq G$ such that for all $a \in \{0,1\}^S$ there exists some $\xi \in X$ such that

$$s \cdot \xi \in U_{a_s} \text{ where } s \in S.$$

This definition leads to the following theorem.

Theorem 3.15 ([KL07]). *A topological dynamical system (X, G) is non-tame if and only if there exists an independence pair.*

Now let $X = \{0,1\}^{\mathbb{Z}}$. Then there is a $\mathbb{Z}$-action on X given by

$$\sigma : (n, x_i) \mapsto x_{i+n}.$$

We refer to σ as *shift* and to $(X, \mathbb{Z})$ as *symbolic dynamical system*. A closed and σ-invariant subset $\Sigma \subseteq X$ is called *subshift*.

Let $a \in \{0,1\}$. By $[a] = \{x \in X \mid x_0 = a\}$ we denote the *cylinder sets of length one* in X. It is not hard to see that, in the case of subshifts, $([0],[1])$ is an independence pair. Thus, we obtain the following.

Corollary 3.16. *Suppose $\Sigma \subseteq \{0,1\}^{\mathbb{Z}}$ is a subshift and there exists an infinite set $S \subseteq \mathbb{Z}$ such that for every $a \in \{0,1\}^S$ there is some $\xi \in \Sigma$ with $\xi_s = a_s$ for all $s \in S$. Then (Σ, σ) is non-tame.*

3.5 Point Spectrum of Dynamical Systems

Suppose (X, G, μ) is a measure-preserving dynamical system, where G is locally compact abelian and second-countable. Recall that the space $L^2(X, \mu)$ of square integrable complex-valued functions on X which is equipped with the inner product

$$\langle f, g \rangle = \langle f, g \rangle_{L^2(X,\mu)} = \int_X \overline{f} g \, \mathrm{d}\mu.$$

A *unitary representation* of G in $L^2(X, \mu)$ is a group homomorphism

$$T : G \to \mathcal{U}(L^2(X, \mu)),$$

where

$$\mathcal{U}(L^2(X,\mu)) = \left\{ U : L^2(X,\mu) \to L^2(X,\mu) \mid U \text{ is surjective, } \langle U(f), U(g) \rangle = \langle f, g \rangle \right\}$$

denotes the space of *unitary operators* of $L^2(X, \mu)$.

The G-action on X induces a unitary representation $T = T_G$ of G on $L^2(X, \mu)$ given by

$$(T^g f)(x) = f(-g \cdot x).$$

Let $\widehat{G}$ be the dual group of G (compare Section 2.1). A non-zero $f \in L^2(X, \mu)$ is called *eigenfunction* of T if there exists an $\chi \in \widehat{G}$ such that

$$T^g f = \chi(g) f$$

for every $g \in G$. Let

$$\mathcal{H}_{pp}(T) = \mathrm{cl}_{L^2(X,\mu)} \left(\mathrm{span}\{ f \in L^2(X,\mu) \mid f \text{ is eigenfunction of } T \} \right).$$

We say T has *pure point spectrum* if $\mathcal{H}_{pp}(T) = L^2(X, \mu)$. In other words, T has pure point spectrum if there exists an orthonormal basis of $L^2(X, \mu)$ which consists of eigenfunctions of T. In either case, we say (X, G, μ) has *pure point spectrum*.

Now let (X, G) be a minimal equicontinuous topological dynamical system. Due to Lemma 3.5 and Theorem 3.12 we may assume without loss of generality that $X = (X, \odot)$ is a compact abelian group such that $g \cdot x = x \odot \phi(g)$ for all $x \in X$ and $g \in G$, where $\phi : G \to X$ denotes a group homomorphism with dense image in X. Observe that we have

$$\chi(x \odot \phi(g)) = \chi(\phi(g))\chi(x)$$

for all $x \in X$, $g \in G$ and $\chi \in \widehat{X}$. Moreover, $\chi \circ \phi \in \widehat{X}$. Then, by the observations above, we obtain

$$(T^g\chi)(x) = \chi(-g \cdot x) = \chi(x \odot \phi(-g)) = \chi(\phi(-g))\chi(x)$$

for every $\chi \in \widehat{X}$. Hence, every character of X is an eigenfunction of (X, G). Now, by the Peter-Weyl Theorem 2.5, the eigenfunctions of (X, G) form an orthonormal basis of $L^2(X, \mu)$.

Hence, we have proven the following well-known fact.

Proposition 3.17. *Let (X, G) be a minimal equicontinuous topological dynamical system. Then (X, G) has pure point spectrum with continuous eigenfunctions.*

3.6 Topological Entropy

In the following we introduce the notion of topological entropy and discuss some of its crucial properties. Suppose (X, d) is a compact metric space and (X, G) a topological dynamical system. We denote the G-action X by φ. Further, let $(A_n)_{n\in\mathbb{N}}$ be a van Hove sequence in G. In all the following definitions we keep the dependence of $(A_n)_{n\in\mathbb{N}}$ implicit.

We say a set $S \subseteq X$ is (ε, n)*-spanning for a set* $K \subseteq X$ if for every $\zeta \in K$ there is some $\xi \in S$ such that

$$\max_{s\in A_n} d(s \cdot \zeta, s \cdot \xi) < \varepsilon.$$

holds. For compact $K \subseteq X$, we denote the minimal cardinality of a set which (ε, n)-spans K by $S^K(\varphi, \varepsilon, n)$.

We say a set $E \subseteq X$ is (ε, n)*-separated* if for all distinct $\zeta, \xi \in E$ holds

$$\max_{s\in A_n} d(s \cdot \zeta, s \cdot \xi) \geq \varepsilon.$$

Let $K \subseteq X$ be compact. By $N^K(\varphi, \varepsilon, n)$ we denote the largest cardinality of any (ε, n)-separated set E contained in K.

The *topological entropy of G on* K is then defined as

$$h^K_{\text{top}}(\varphi) = \lim_{\varepsilon\to 0} h^K_\varepsilon(\varphi),$$

where

$$\begin{aligned} h^K_\varepsilon(\varphi) &= \limsup_{n\to\infty} \frac{1}{\Theta_G(A_n)} \log S^K(\varphi, \varepsilon, n) \\ &= \limsup_{n\to\infty} \frac{1}{\Theta_G(A_n)} \log N^K(\varphi, \varepsilon, n). \end{aligned}$$

We set $h_{\text{top}}(\varphi) = h^X_{\text{top}}(\varphi)$.

Now let ψ denote another G-action on a compact metric space Y. Suppose (Y, ψ) is a factor of (X, φ) with factor map $\beta : X \to Y$. Then we can relate the entropy of the systems. Indeed, we have

$$h_{\text{top}}(\psi) \leq h_{\text{top}}(\varphi)$$

(compare [KH97]). For $\xi \in Y$, let $h^{\xi}_{\text{top}}(\varphi) = h^{\beta^{-1}(\xi)}_{\text{top}}(\varphi)$. Clearly, we then have $h^{\xi}_{\text{top}}(\varphi) \leq h_{\text{top}}(\varphi)$ for any $\xi \in Y$. In case of $G = \mathbb{R}$ we then can formulate

Theorem 3.18 ([Bow71]). *Let $(X, \mathbb{R})$ be a topological dynamical system, where the $\mathbb{R}$-action is denoted by φ. Suppose $(Y, \mathbb{R})$ is a factor of $(X, \mathbb{R})$ with factor map $\beta : X \to Y$ and denote by ψ the $\mathbb{R}$-action on Y. Then we have*

$$h_{\text{top}}(\varphi) \leq \varphi(\psi) + \sup_{\xi \in Y} h^{\xi}_{\text{top}}(\varphi).$$

In case of $h_{\text{top}}(\psi) = 0$, the preceding inequalities yields $h_{\text{top}}(\varphi) = \sup_{\xi \in Y} h^{\xi}_{\text{top}}(\varphi)$, which means that the entropy is realised already in a single fibre of β. Note also that, in the Euclidean case, vanishing entropy with respect to a van Hove sequence implies vanishing entropy with respect to all van Hove sequences (compare [BLR07]).

3.7 Ergodic Theorems for Abelian Group Actions and the Lattice Counting Problem

Let X be a compact metric space and consider the discrete measure-preserving dynamical system $(X, \mathbb{Z}, \mu)$. Recall that the $\mathbb{Z}$-action on X might be represented via some homeomorphism $T : X \to X$. The following theorems are well-known.

Theorem 3.19 (Birkhoff's Ergodic Theorem). *Assume $(X, \mathbb{Z}, \mu)$ is an ergodic dynamical system. Let $f : X \to \mathbb{R}$ be a measurable function. Then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(T^i(x)) = \int_X f \, \mathrm{d}\mu$$

for μ-almost every $x \in X$.

Theorem 3.20. *Assume $(X, \mathbb{Z}, \mu)$ is an uniquely ergodic dynamical system. Let $f : X \to \mathbb{R}$ be a continuous function. Then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(T^i(x)) = \int_X f \, \mathrm{d}\mu$$

uniformly for every $x \in X$.

It is natural to ask whether a similar statement holds for general group actions or not. In fact, a main difficulty comprises of choosing a sequence along which we take the averages of the sum on the left above. However, if the sequence is a chosen to be a Følner sequence (compare Section 2.2) and the group acting on X is nice enough in a certain sense, we acquire a similar result in greater generality.

Let G be an abelian group and suppose $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence in G (compare Section 2.2). Define the value

$$I_n(x, f) = \frac{1}{\Theta_G(F_n)} \int_{F_n} f(g \cdot x) \, \mathrm{d}\Theta_G(g) \tag{3.7.1}$$

for $x \in X$ and $f : X \to \mathbb{R}$.

Theorem 3.21 (Pointwise Ergodic Theorem for Group Actions, [MR13])**.** *Let X be a compact metrizable space and G a locally compact abelian, second-countable group acting measurably on X. Assume that $(F_n)_{n\in\mathbb{N}}$ is a tempered Følner sequence in G. Let μ be a G-invariant probability measure on X and let $f \in L^1(X,\mu)$ be given. Then $I_n(x,f)$ exists and is finite for μ-almost every $x \in X$ and all $n \in \mathbb{N}$. Furthermore, there exists a G-invariant function $\tilde{f} \in L^1(X,\mu)$ such that*

$$\int_X \tilde{f} \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$$

and

$$\lim_{n\to\infty} I_n(x,f) = \tilde{f}(x)$$

for μ-almost every $x \in X$.

Moreover, the following statements are equivalent:

(i) The measure μ is ergodic.

(ii) For every $f \in L^1(X,\mu)$ we have

$$\lim_{n\to\infty} I_n(x,f) = \int_X f \, \mathrm{d}\mu$$

for μ-almost every $x \in X$.

(iii) There exists a dense subset $\mathcal{F} \subseteq C(X)$ such that for every $f \in \mathcal{F}$, we have

$$\lim_{n\to\infty} I_n(x,f) = \int_X f \, \mathrm{d}\mu$$

for μ-almost every $x \in X$.

Remark 3.22. Note that the limit in (ii) and (iii) is independent of the choice of the tempered Følner sequence.

In case of uniquely ergodic systems and under assumption of continuous functions f, we may exclude the exceptional set of zero measure in the previous theorem. Note that the following theorem holds under more general assumptions for the acting group G.

Theorem 3.23 ([MR13])**.** *Let X be compact metrizable space and G a locally compact and abelian group. Let $(F_n)_{n\in\mathbb{N}}$ be a Følner sequence in G. Then the following are equivalent:*

(i) There exists exactly one G-invariant measure μ on X.

(ii) For every $f \in C(X)$ there exists a constant $I(f)$ such that the limit

$$\lim_{n\to\infty} I(x,f) = I(f)$$

exists uniformly for all $x \in X$.

(iii) There exists a dense subset $\mathcal{F} \subseteq C(X)$ such that for every $f \in \mathcal{F}$ there exists a constant $I(f)$ such that pointwise for every $x \in X$ we have

$$\lim_{n\to\infty} I(x,f) = I(f).$$

In either case, the measure μ is ergodic and the above statements hold with $I(f) = \int_X f \, \mathrm{d}\mu$.

Remark 3.24 ([MR13])**.** The limits above are independent of the choice of the Følner sequence. Also, in comparison to the Pointwise Ergodic Theorem 3.21, the Følner sequence is not required to fulfil additional conditions like being tempered.

Remark 3.25. We want to point out that both theorems above might be formulated for non-abelian groups acting on X.

Let (X, G, μ) be an ergodic dynamical system. In the remaining section, we assume G is a countable, discrete and abelian group. We say a point $x \in X$ is *generic* if for every $f \in C(X)$ there exists a constant $I(f)$ such that $\lim_{n\to\infty} I(x, f) = I(f)$ exists uniformly. As an immediate consequence of Theorem 3.23 we then obtain

Corollary 3.26. *If (X, G, μ) is uniquely ergodic, every point $x \in X$ is generic.*

As seen in the following lemma, it is possible to extend the theorem above to characteristic functions.

Lemma 3.27. *Suppose (X, G, μ) is uniquely ergodic and let $W \subseteq X$ such that $\operatorname{int}(W) \neq \emptyset$ and $\mu(\partial W) = 0$. Then $\lim_{n\to\infty} I_n(x, \chi_W) = \mu(W)$ uniformly for all $x \in X$.*

Proof. Choose an open set $U \subseteq X$. Since the system is uniquely ergodic, every point $x \in X$ is generic. By Urysohn's Lemma we can find a sequence $f_k \in C_c(X)$ such that $f_k \nearrow \chi_U$. Hence, $f_k \leq \chi_U$, which leads to

$$\liminf_{n\to\infty} I_n(x, \chi_U) \geq \lim_{n\to\infty} I_n(x, f_k) = \int_X f_k \,\mathrm{d}\mu \to \mu(U).$$

In a similar way we obtain

$$\limsup_{n\to\infty} I_n(x, \chi_C) \leq \mu(C)$$

for any closed $C \subseteq X$.

Choosing $U = \operatorname{int}(W)$ and $C = \operatorname{cl}(W)$ yields $\chi_U \leq \chi_W \leq \chi_C$. Hence,

$$\liminf_{n\to\infty} I_n(x, \chi_W) \geq \mu(U) \text{ and } \limsup_{n\to\infty} I_n(x, \chi_W) \leq \mu(C).$$

Since $\mu(\partial W) = 0$ we obtain

$$\mu(W) = \mu(U) \leq \liminf_{n\to\infty} I_n(x, \chi_W) \leq \limsup_{n\to\infty} I_n(x, \chi_W) \leq \mu(C) = \mu(W)$$

which implies $\lim_{n\to\infty} I_n(x, \chi_W) = \mu(W)$ for all $x \in X$. □

Now let $f \in L^1(X, \mu)$. We say $x \in X$ is *generic with respect to f* if there exists a constant $I(f)$ such that $\lim_{n\to\infty} I(x, f) = I(f)$ uniformly. The above lemma ensures that, in the uniquely ergodic case, all points $x \in X$ are generic with respect to a characteristic function χ_W as long as the set W is "nice enough". A direct consequence of this is the following lemma:

Lemma 3.28. *Let (X, G, μ) be uniquely ergodic. Suppose $\mathcal{P} = \{X_1, \dots, X_N\}$ is a partition of X such that $\mu(\partial X_j) = 0$ for all $j = 1, \dots, N$. Let $a_j \in \mathbb{C}^d$, $j = 1, \dots, N$, and consider the function*

$$f : X \to \mathbb{C}^d : x \mapsto \sum_{j=1}^{N} a_j \chi_{X_j}(x).$$

Then all $x \in X$ are generic with respect to f.

Proof. Let (F_n) be Følner sequence in G. By Lemma 3.27 we have

$$\begin{aligned}
\lim_{n\to\infty} I_n(x, f) &= \lim_{n\to\infty} \frac{1}{\Theta_G(F_n)} \sum_{g\in F_n} \sum_{j=1}^{N} a_j \chi_{X_j}(g \cdot x) \\
&= \sum_{j=1}^{N} a_j \lim_{n\to\infty} \left(\frac{1}{\Theta_G(F_n)} \sum_{g\in F_n} \chi_{X_j}(g \cdot x) \right) \\
&= \sum_{j=1}^{N} a_j \mu(X_j) = \int_X f \mathrm{d}\mu.
\end{aligned}$$

□

Remark 3.29. A *partition* in this context is a finite collection $\{P_1, \dots, P_N\}$ of subsets of X with non-empty interior such that $\bigcup_{j=1}^N P_j = X$ and $\operatorname{int}(P_i) \cap \operatorname{int}(P_j) = \emptyset$ for all $i \neq j$.

The remaining part of this section is, in a certain sense, a continuation of Chapter 2. We suppose T is a lattice in a second-countable lca group G. Then it seems natural to ask what the relation between $\Theta_T(A_n \cap T)$ and $\Theta_G(A_n)$ is for a given sequence of subsets $A_n \subseteq G$. In other words, we ask for the existence of some constant $\kappa(T, (A_n)) > 0$ such that

$$\lim_{n\to\infty} \frac{\Theta_T(A_n \cap T)}{\Theta_G(A_n)} = \kappa(T, (A_n)).$$

This problem is also known as the *lattice counting problem* and, for instance, discussed in [ME93], [DRS93] as well as [GN12]. Unsurprisingly, it turns out that answers to this problem heavily depend on properties of the sequence $(A_n)_{n\in\mathbb{N}}$.

In [GN12], the lattice counting problem was discussed in the (broader) setting of second-countable locally compact groups. To tackle this problem, the authors introduced a more general class of sequences of sets. We say a sequence of non-empty sets $B_n \subseteq G$ is *well-rounded* if for any $\varepsilon > 0$ there exists an open neighbourhood U of the identity in G such that

$$\frac{\Theta_G(U + \partial B_n)}{\Theta_G(B_n)} < \varepsilon$$

for all sufficiently large $n \in \mathbb{N}$. It turns out that van Hove sequences are well-rounded:

Lemma 3.30. *Let G be a lca group and $(A_n)_{n\in\mathbb{N}}$ a van Hove sequence in G. Then $(A_n)_{n\in\mathbb{N}}$ is well-rounded.*

Proof. Observe that, by definition of the K-boundary, for every $n \in \mathbb{N}$ and compact neighbourhood of the origin $K \subseteq G$ we obtain $K + \partial A_n \subseteq \partial^K A_n$. Let $\varepsilon > 0$. Since $(A_n)_{n\in\mathbb{N}}$ is van Hove we can choose a neighbourhood U of 0 such that $\frac{\Theta_G(\partial^{\mathrm{cl}(U)} A_n)}{\Theta_G(A_n)} < \varepsilon$ for sufficiently large $n \in \mathbb{N}$. Thus,

$$\frac{\Theta_G(U + \partial A_n)}{\Theta_G(A_n)} \leq \frac{\Theta_G(\partial^{\mathrm{cl}(U)} A_n)}{\Theta_G(A_n)} < \varepsilon.$$

Hence, $(A_n)_{n\in\mathbb{N}}$ is well-rounded. ☐

The following theorem provides an answer to the lattice point counting problem.

Theorem 3.31 (Lattice Point Counting Theorem, [GN12])**.** *Let G be a second-countable lca group and $T \leq G$ a lattice. Suppose $(A_n)_{n\in\mathbb{N}}$ is a well-rounded sequence in G such that Theorem 3.23 holds along $(A_n)_{n\in\mathbb{N}}$. Then we have*

$$\lim_{n\to\infty} \frac{\Theta_T(A_n \cap T)}{\Theta_G(A_n)} = \frac{1}{\mu(G/T)}, \tag{3.7.2}$$

where μ denotes the measure of a fundamental domain of T in G with respect to a fixed choice of Θ_G.

Remark 3.32. (i) Note that in [GN12], a sequence $(A_n)_{n\in\mathbb{N}}$ is called well-rounded if for every $\delta > 0$ there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$\Theta_G(B_\varepsilon(0) + A_n) \leq (1+\delta)\Theta_G\left(\bigcap_{g\in B_\varepsilon(0)} g + A_n\right)$$

holds for all $n \geq N$. However, it is not hard to see that our definition of well-roundedness implies this condition (compare also [ME93] for the case of affine symmetric spaces).

(ii) The original statement given in [GN12] needs less assumptions then our statement. In particular, the groups involved there are neither assumed to be abelian nor amenable. Thus, it is not clear whether such groups admit Følner sequences which is the reason for introducing terms like being well-rounded. Furthermore, those well-rounded sequences had to be chosen such that a *mean ergodic theorem* holds, that is, (3.7.1) converges in L^1-norm for all $f \in L^1(G/T)$.

However, we want to point out, that, according to our previous discussions, each van Hove sequence $(A_n)_{n\in\mathbb{N}}$ is well-rounded and Følner. Further, if Theorem 3.23 holds along such a sequence, the mean ergodic theorem is also satisfied (compare also [Lin99]). Since G/T admits a unique G-invariant measure, the assumptions of Theorem 3.23 are met and hence (3.7.2) holds along all van Hove sequences. Therefore, in our context it is sufficient to assume that $(A_n)_{n\in\mathbb{N}}$ is a van Hove sequence instead of assuming well-roundedness and validity of a mean ergodic theorem along $(A_n)_{n\in\mathbb{N}}$.

As a preparation for later discussions, we will conclude this chapter with the following general lemma regarding preimages of van Hove sequences.

Lemma 3.33. *Let G be a lca group and assume $T \leq G$ is a lattice. Suppose $(A_n)_{n\in\mathbb{N}}$ is a van Hove sequence in G and let $H : T \to G$ be an injective homomorphism. Then $(H^{-1}(A_n))_{n\in\mathbb{N}}$ is a van Hove sequence in T.*

Proof. Let $\Gamma = H(T)$. Then $\Gamma \cong T/\ker(H)$. Since H is injective, $\ker(H)$ is trivial. Thus, Γ is also a lattice in G. We define $F_n = H^{-1}(A_n)$. Clearly, each F_n is compact and we obtain $\bigcup_{n\in\mathbb{N}} F_n = T$ as well as $F_{n-1} \subseteq F_n$. Now suppose that $K \subseteq T$ is compact. Without loss of generality we assume that $0 \in K$. Using $\mathrm{cl}(H^{-1}(A_n)^c) = \mathrm{cl}(H^{-1}(A_n^c)) \subseteq H^{-1}(\mathrm{cl}(A_n^c))$ and $H^{-1}(\mathrm{int}(A_n)) \subseteq \mathrm{int}(H^{-1}(A_n))$ as well as $H^{-1}(A_n) + K \subseteq H^{-1}(A_n + H(K))$, a straightforward calculations shows $\partial^K F_n \subseteq H^{-1}\left(\partial^{H(K)} A_n\right)$.

Note that we have

$$\Theta_T(F) = \sharp(H^{-1}(A) \cap T) = \sharp(A \cap \Gamma) = \Theta_\Gamma(A \cap \Gamma)$$

for any set $A \subseteq G$ with $H^{-1}(A) = F \subseteq T$.

Apart from that, by Lemma 2.11, both $B_n = A_n + H(K)$ and $C_n = A_n \setminus \partial^{H(K)} A_n$ are van Hove sequences in G. In particular, Theorem 3.31 holds for those sequences, i.e., $\Theta_\Gamma(B_n \cap \Gamma) = \frac{1}{\mu(G/\Gamma)} \Theta_G(B_n) + o(1)$ as well as $\Theta_\Gamma(C_n \cap \Gamma) = \frac{1}{\mu(G/\Gamma)} \Theta_G(C_n) + o(1)$ as $n \to \infty$. This directly yields that

$$\Theta_\Gamma(\partial^{H(K)} A_n \cap \Gamma) = \frac{1}{\mu(G/\Gamma)} \Theta_G(\partial^{H(K)} A_n) + o(1) \text{ as } n \to \infty. \tag{3.7.3}$$

This leads to

$$\begin{aligned} \frac{\Theta_T(\partial^K F_n)}{\Theta_T(F_n)} \leq \frac{\Theta_T(H^{-1}(\partial^{H(K)} A_n))}{\Theta_T(H^{-1}(A_n))} &= \frac{\Theta_\Gamma(\partial^{H(K)} A_n \cap \Gamma)}{\Theta_\Gamma(A_n \cap \Gamma)} \\ &= \frac{\Theta_\Gamma(\partial^{H(K)} A_n \cap \Gamma)}{\Theta_G(\partial^{H(K)} A_n)} \cdot \frac{\Theta_G(A_n)}{\Theta_\Gamma(A_n \cap \Gamma)} \cdot \frac{\Theta_G(\partial^{H(K)} A_n)}{\Theta_G(A_n)}. \end{aligned}$$

Now Equation (3.7.3), Lemma 3.30 and Theorem 3.31 yield

$$\lim_{n\to\infty} \frac{\Theta_\Gamma(\partial^{H(K)} A_n \cap \Gamma)}{\Theta_G(\partial^{H(K)} A_n)} = \frac{1}{\mu(G/T)}$$

as well as

$$\lim_{n\to\infty} \frac{\Theta_G(A_n)}{\Theta_\Gamma(A_n \cap \Gamma)} = \mu(G/T).$$

Together with the van Hove property of $(A_n)_{n\in\mathbb{N}}$ this shows that

$$\lim_{n\to\infty} \frac{\Theta_T(\partial^K F_n)}{\Theta_T(F_n)} \leq \lim_{n\to\infty} \frac{\Theta_G(\partial^{H(K)} A_n)}{\Theta_G(A_n)} = 0.$$

Hence, F_n is a van Hove sequence in T. □

Chapter 4

Aperiodic Order

4.1 Aperiodic Sets

Let $G = (G, +)$ be a locally compact abelian second-countable group with identity 0. Note that by Lemma 2.2 G admits a proper G-invariant metric d and G is σ-compact. A set $\Gamma \subseteq G$ is called *(r)-uniformly discrete* if there exists $r > 0$ such that

$$d(g, h) \geq r \text{ for all distinct } g, h \in G.$$

We call $\Gamma \subseteq G$ *(R)-relatively dense* if there exists $R > 0$ such that

$$\Gamma \cap B_R(g) \neq \emptyset \text{ for all } g \in G.$$

A set $\Gamma \subseteq G$ is called *(r-R)-Delone* if it is both uniformly discrete and relatively dense.

Remark 4.1. (i) By σ-compactness of G, every uniformly discrete set in G has to be countable.

(ii) Note that "relatively dense" and "syndetic" (as defined in 3.1) describe actually the same property. In the context of point sets we are going to use the term "relatively dense", whereas in the context of dynamical systems we will use the term "syndetic".

(iii) In fact, Delone sets can also be introduced for non-second-countable groups. In this case, the terms above are defined with respect to the topology on G. However, since we won't lose much generality by restricting ourselves to the second-countable case, we will stick to this assumption.

We say a uniformly discrete set $\Gamma \subseteq G$ is *aperiodic* if $\Gamma - g = \Gamma$ implies $g = 0$.

A set $\Gamma \subseteq G$ has *finite local complexity (FLC)* if

$$\sharp\{(\Gamma - g) \cap B_R(0) \mid g \in \Gamma\} < \infty$$

for all $R > 0$.

Lemma 4.2 ([Sch99]). *A set $\Gamma \subseteq G$ has FLC if and only if $\Gamma - \Gamma$ is closed and discrete.*

We say a Delone set $\Gamma \subseteq G$ is *Meyer* if there exists a finite set $F \subseteq G$ such that

$$\Gamma - \Gamma \subseteq \Gamma + F.$$

Lemma 4.3 ([Lag98], [BLM07]). *Let $\Gamma \subseteq G$ be Delone.*

(i) If Γ is Meyer then $\Gamma - \Gamma$ is uniformly discrete.

(ii) Suppose G is compactly generated. Then Γ is Meyer if and only if $\Gamma - \Gamma$ is uniformly discrete.

The connection between the terms defined above is seen in the following easy corollary.

Corollary 4.4. *Let $\Gamma \subseteq G$. Then the following implications are strict:*

$$\Gamma \textit{ is Meyer } \Rightarrow \Gamma \textit{ has (FLC) and is Delone } \Rightarrow \Gamma \textit{ is Delone.}$$

Given a set $\Gamma \subseteq G$ and $g \in \Gamma, R > 0$ the pair $(P(R, g), R)$ with

$$P(R, g) = (\Gamma - g) \cap B_R(0)$$

is called a *(R)-patch of Γ in g*. The set of all patches of Γ is denoted by

$$\mathcal{P}(\Gamma) = \{(P(R, g), R) \mid R > 0, g \in \Gamma\}.$$

Note that Γ is not required to be Delone for this definition. In case of $G = \mathbb{R}^n$ it is possible to characterize the FLC-property with patches.

Lemma 4.5 ([Lag98])**.** *Let $\Gamma \subseteq \mathbb{R}^n$ be a Delone set such that $\mathbb{R}^n = \bigcup_{x\in\Gamma} B_R(x)$. Then the following are equivalent:*

(i) Γ *has FLC.*

(ii) $\sharp\{(\Gamma - x) \cap B_{2R}(0) \mid x \in \Gamma\} < \infty$.

Let $\Gamma \subseteq G$ be a set with FLC. We say Γ is *repetitive* if for all $(P, R) \in \mathcal{P}(\Gamma)$ the set

$$\{g \in \Gamma \mid P(R, g) = P\}$$

is syndetic. For given van Hove sequence $(A_n)_{n\in\mathbb{N}}$ in G and patch $(P, R) \in \mathcal{P}(\Gamma)$, if the limit

$$\nu(P, \Gamma, (A_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \sharp\{l \in \Gamma \cap A_n \mid P(R, l) = P\}$$

exists, we call it *patch frequency* of P. We say Γ has *uniform patch frequencies along* $(A_n)_{n\in\mathbb{N}}$ if for all $(P, R) \in \mathcal{P}(\Gamma)$ the limit

$$\lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \sharp\{l \in (\Gamma - g) \cap A_n \mid P(R, l) = P\}$$

exists and the convergence is uniform in $g \in G$. If the above holds along every van Hove sequence $(A_n)_{n\in\mathbb{N}}$ in G we say Γ has *uniform patch frequencies (UPF)*.

Remark 4.6. If Γ has UPF the limit does not depend on the choice of the sequence $(A_n)_{n\in\mathbb{N}}$ as long as it is a van Hove sequence (compare [Sch99]).

4.2 Cut and Project Schemes

A well-known approach to create Delone sets is to make use of cut and project schemes. In this section we will introduce and discuss this method.

A *(classical) Cut and Project Scheme (CPS)* is a triple $(G, H, \mathcal{L})$ such that

(i) G and H are lca groups,

(ii) G is σ-compact,

(iii) $\mathcal{L} \subseteq G \times H$ is a lattice such that the canonical projections $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ satisfy

   (1) the restriction $\pi_G|_{\mathcal{L}}$ is injective,

   (2) $\pi_H(\mathcal{L})$ is dense in H.

The space G is also called *external group* or *physical group* and the space H is called *internal group*. As before, we assume all involved groups are second-countable. Thus, G is automatically σ-compact and we obtain metrics on G and H. We set $L = \pi_G(\mathcal{L})$ (sometimes called *structure group*) and $L^* = \pi_H(\mathcal{L})$. Injectivity of $\pi_G|_{\mathcal{L}}$ then yields the existence of a well-defined map

$$* : L \to L^* : l \mapsto l^* = \pi_H(\pi_G^{-1}(l)).$$

It is not hard to see that $*$ is in fact a homomorphism. We refer to $*$ as *star map*. We may visualize the components of CPS in the following scheme:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_G} & G \times H & \xrightarrow{\pi_H} & H \\ \cup & & \cup & & \cup \\ L & \xleftarrow{1-1} & \mathcal{L} & \xrightarrow{\text{dense}} & L^* \end{array}$$

Given a subset $W \subseteq H$ (in this context W is called *window*), one obtains a point set

$$⅄(W) = \pi_G(\mathcal{L} \cap (G \times W)).$$

Lemma 4.7 ([Rob07]). *Let $(G, H, \mathcal{L})$ be a CPS with window W.*

(i) If W is compact, then ⅄(W) is uniformly discrete.

(ii) If $\mathrm{int}(W) \neq \emptyset$, then ⅄$(W)$ is relatively dense.

If the window $W \subseteq H$ is compact, then we call ⅄(W) and all translates ⅄$(W) - g$, $g \in G$, *weak model set*. We say a window $W \subseteq H$ is *proper* if $W = \mathrm{cl}(\mathrm{int}(W))$. In this case ⅄$(W)$ is Delone and we refer to ⅄(W) and all its translates ⅄$(W) - g$, $g \in G$, as *model set*.

Lemma 4.8 ([Rob07]). *Let $(G, H, \mathcal{L})$ be a CPS with proper window W. Then ⅄(W) is Meyer.*

On the converse, each Meyer set is a subset of some model set.

Lemma 4.9 ([ABKL15]). *Let $\Lambda \subseteq G$ be a Meyer set. Then there exists a CPS $(G, H, \mathcal{L})$ with compact window $W \subseteq H$ such that $\Lambda \subseteq$ ⅄(W).*

In the case of weak model sets, ⅄(W) is not guaranteed to be relatively dense. However, ⅄(W) inherits still more structure then only being uniformly discrete.

Lemma 4.10 ([HR15]). *Let $(G, H, \mathcal{L})$ be a CPS with compact window W. Then ⅄(W) has FLC.*

We call the window W *generic* if $\partial W \cap L^* = \emptyset$. The window is called *regular* if $\Theta_H(\partial W) = 0$, otherwise we refer to the window as *irregular*. In all cases we may refer to ⅄(W) itself as *generic* or *(ir)regular*. The following result ensures that any given window might be translated to a generic window.

Lemma 4.11 ([Sch99]). *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. Then there exists $h \in H$ such that $W + h$ is generic.*

Proof. Since $\mathcal{L}$ is countable and $\mathrm{int}(\partial W) = \emptyset$,

$$L^* - \partial W = \bigcup_{l \in L} (l^* - \partial W)$$

cannot agree with H by Baire's category theorem. Then any $h \in H \setminus (L^* - \partial W)$ has the desired property. $\square$

Now we will see that the above introduced additional properties of the window determine geometrical properties of the corresponding model set.

Lemma 4.12 ([Sch99]). *Let $(G, H, \mathcal{L})$ be a CPS with proper window W.*

(i) If W is generic, then ⅄(W) is repetitive.

(ii) If W is regular, then $\curlywedge(W)$ has UPF.

Finally, we want to discuss the special case $G = \mathbb{R}^N$. We will call such a CPS $(\mathbb{R}^N, H, \mathcal{L})$ *Euclidean CPS*. In this situation we will sometimes refer to $\mathcal{L}$ as *irrational lattice*. In case $G = H = \mathbb{R}$ we refer to the CPS as *planar*.

Since densities of model sets in Euclidean spaces will play an important role in our later considerations, we want to point out some relations between the structure of the window $W \subseteq H$, the lattice $\mathcal{L}$ and the density of the corresponding model set $\curlywedge(W)$ for given CPS $(\mathbb{R}^N, H, \mathcal{L})$. For this, we use the partial order on $\mathbb{R}^N$ given by

$$s \leq t \Longleftrightarrow s_i \leq t_i \text{ for all } i = 1, \ldots, N.$$

Given $n \in \mathbb{N}$, we let $\overline{n} = (n, \ldots, n) \in \mathbb{R}^N$ and denote by

$$A_n = \{s \in \mathbb{R}^N \mid -\overline{n} \leq s \leq \overline{n}\}$$

the cubes of sidelength $2n$ and volume $(2n)^N$. Note that $(A_n)_{n\in\mathbb{N}}$ is a tempered van Hove sequence. Suppose $\Gamma \subseteq \mathbb{R}^N$ is a uniformly discrete set. Then we define its *asymptotic density* by

$$\nu_\Gamma = \limsup_{n\to\infty} \frac{\sharp(\Gamma \cap A_n)}{\mathrm{Leb}(A_n)}.$$

If ν_Γ is actually a limit, we call it the *density* of Γ.

Theorem 4.13 (Density Formula for Model Sets, [Moo02]). *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with measurable window $W \subseteq H$ and let $\mu(\mathcal{L})$ denote the measure of a fundamental domain of $(\mathbb{R}^N \times H)/\mathcal{L}$. Then the following holds.*

(i) For Θ_H-almost every $h \in H$ the density of $\curlywedge(W + h)$ is given by

$$\nu_{\curlywedge(W+h)} = \frac{\Theta_H(W)}{\mu(\mathcal{L})}.$$

If $\Theta_H(\partial W) = 0$ then the statement holds for all $h \in H$.

(ii) If W is compact, the inequality

$$\limsup_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sharp(\curlywedge(W + h) \cap A_n) \leq \frac{\Theta_H(W)}{\mu(\mathcal{L})}$$

holds for all $h \in H$.

(iii) If W is open, the inequality

$$\liminf_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sharp(\curlywedge(W + h) \cap A_n) \geq \frac{\Theta_H(W)}{\mu(\mathcal{L})}$$

holds for all $h \in H$.

Proof. Part (i) of the theorem is shown in [Moo02]. Inspecting the proof there one can easily infer part (ii) and (iii) as well. For convenience we nevertheless want to sketch a proof. Let $\mathbb{T} = (\mathbb{R}^N \times H)/\mathcal{L}$ be equipped with a natural $\mathbb{R}^N$-action ω (compare the beginning of Section 4.4 for a discussion). Let $\sigma : \mathbb{R}^N \to [0, \infty)$ be a continuous function with compact support and $\int_{R^N} \sigma \,\mathrm{dLeb} = 1$. We define the function

$$f : \mathbb{T} \to [0, \infty) : \xi \mapsto \sum_{(s,h)\in-\xi} \sigma(s)\chi_W(h).$$

Note that $\xi = (s, h) + \mathcal{L} \in \mathbb{T}$ denotes an equivalence class (see Section 4.4). Since both σ and χ_W vanish outside compact sets the sum has only finitely many non-vanishing terms and is thus measurable.

Define for $\xi = [s,h]_{\mathcal{L}}$ the set $\curlywedge(\xi) = \curlywedge(W+h) - s$. Then a short computation (compare [Moo02]) shows that

$$\left| \frac{1}{\mathrm{Leb}(A_n)} \int_{A_n} f(\omega_s(\xi)) \,\mathrm{dLeb} - \frac{1}{\mathrm{Leb}(A_n)} \sharp(\curlywedge(\xi) \cap A_n) \right| \to 0 \text{ as } n \to \infty$$

for all $\xi \in \mathbb{T}$. Then the desired statements (i), (ii) and (iii) follow from the corresponding statements for the averages

$$\alpha_n(\xi) = \frac{1}{\mathrm{Leb}(A_n)} \int_{A_n} f(\omega_s(\xi)) \,\mathrm{dLeb}.$$

These statements in turn hold as $(\mathbb{T}, \mathbb{R}^N)$ is uniquely ergodic.

(i). Birkhoff's Ergodic Theorem 3.19 directly implies convergence of $\alpha_n(\xi)$ for almost every $\xi \in \mathbb{T}$. As convergence for ξ implies convergence for all $\omega_s(\xi)$, $s \in \mathbb{R}^N$, the almost sure convergence in $\xi \in \mathbb{T}$ implies almost-sure convergence in $h \in H$. Moreover, Theorem 3.20 implies the second statement.

(ii). By replacing χ_W with a continuous function with compact support, we obtain continuity of f and uniform convergence in all $\xi \in \mathbb{T}$ by Oxtoby's Theorem (compare [Oxt52]). Approximating χ_W from above by continuous functions with compact support we obtain statement (ii) uniformly in $\xi \in \mathbb{T}$ and thus also in $h \in H$.

(iii). We replace the approximation used in the proof of (ii) above by an approximation from below. By regularity of Θ_H for each $\varepsilon > 0$ we may choose a compact set $K \subseteq W$ such that $\Theta_H(W) = \Theta_H(K) + \varepsilon$. Now, invoking Urysohns Lemma we can choose a function f with compact support and $\chi_K \leq f \leq \chi_W$ (compare also the proof of Lemma 3.27, where similar arguments are used). ☐

Remark 4.14. We want to mention that the original theorem in [Moo02] does not restrict to Euclidean spaces. However, if $\Theta_H(\partial W) > 0$, one has to take care about the properties of the averaging sequences. Since we will only use the density formula for $G = \mathbb{R}^N$, to avoid technicalities, we stick to the formulation given above.

Now we assume additionally that $H = \mathbb{R}^M$. In the following, we want to give two statement regarding the lattice involved in such Cut and Project Schemes.

Lemma 4.15 ([HR15])**.** *Let $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$ be a CPS. Then for any open $U \subseteq \mathbb{R}^M$ there exists a compact $F \subseteq G$ satisfying $(F \times U) + \mathcal{L} = \mathbb{R}^{N+M}$.*

Remark 4.16. Basically, this lemma states that denseness of L^* implies the existence of "arbitrarily thin" fundamental domains of $\mathcal{L}$. Recall that fundamental domains of lattices always have the same volume. Thus, by varying the set U, one might find arbitrarily many primitive vectors of the lattice $\mathcal{L}$.

Finally, we want to study the properties of matrices which generate lattices belonging to Euclidean CPS. We consider the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1,N+M} \\ \vdots & & \vdots \\ a_{N+M,1} & \cdots & a_{N+M,N+M} \end{pmatrix} \in \mathrm{GL}(N+M, \mathbb{R}).$$

Put $v_j = \begin{pmatrix} a_{1,j} & \dots & a_{N,j} \end{pmatrix}^T \in \mathbb{R}^N$ and $w_i = \begin{pmatrix} a_{N+1,j} & \dots & a_{N+M,j} \end{pmatrix}^T \in \mathbb{R}^M$ for $j = 1, \dots, N+M$. Then the matrix may be written as

$$A = \begin{pmatrix} v_1 & \dots & v_{N+M} \\ w_1 & \dots & w_{N+M} \end{pmatrix}.$$

We say A is an *irrational matrix (with respect to N and M)* if the following properties hold:

(IM1) The vectors $v_1, \dots, v_{N+M}$ are *rationally independent*, i.e., for all $\boldsymbol{k} = (k_1, \dots, k_{N+M}) \in \mathbb{Z}^{N+M}$ holds

$$\sum_{j=1}^{N+M} k_j v_j = 0 \Longrightarrow k_j = 0 \text{ for all } j = 1, \dots, N+M.$$

(IM2) Let $w_{j_1}, \ldots, w_{j_M}$ denote M linearly independent vectors. Then there exists at least one index $j_0 \in \{1, \ldots, N+M\} \setminus \{j_1, \ldots, j_M\}$ such that all entries of w_{j_0} are *rationally independent with respect to* $\begin{pmatrix} w_{j_1} & \ldots & w_{j_M} \end{pmatrix}^{-1}$, i.e., the entries of

$$\begin{pmatrix} w_{j_1} & \ldots & w_{j_M} \end{pmatrix}^{-1} w_{j_0} \in \mathbb{R}^M$$

are rationally independent.

Remark 4.17. Observe that regularity of A ensures that there exist N linearly independent vectors in $\{v_1, \ldots, v_{N+M}\}$ as well as M linearly independent vectors in $\{w_1, \ldots, w_{N+M}\}$.

It turns out, that, for a given Euclidean Cut and Project Scheme with an Euclidean internal space, we may describe the lattice via such an irrational matrix.

Lemma 4.18. *Let* $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$ *be an Euclidean CPS. Then each matrix* A *satisfying* $\mathcal{L} = A(\mathbb{Z}^{N+M})$ *is irrational.*

Proof. Let A be a matrix with $\mathcal{L} = A(\mathbb{Z}^{N+M})$. We may write

$$A = \begin{pmatrix} u_1 & \ldots & u_{N+M} \end{pmatrix},$$

where $u_1, \ldots, u_{N+M} \in \mathbb{R}^{N+M}$. Since $\mathcal{L}$ is a lattice, $\{u_1, \ldots, u_{N+M}\}$ is a set of linearly independent vectors and in particular we have $A \in \mathrm{GL}(N+M, \mathbb{R})$.

Define the vectors v_j and w_j as in the discussion before the lemma. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{N+M}) \in \mathbb{Z}^{N+M}$ be arbitrary. We have to show that

$$\sum_{j=1}^{N+M} v_j \alpha_j = 0 \text{ implies } \alpha_j = 0 \text{ for all } j = 1, \ldots, N+M.$$

To that end, choose $x \in \mathcal{L}$ such that $x = A\boldsymbol{\alpha}$. Then the above sum equals $\pi_{\mathbb{R}^N}(x)$. Due to injectivity of $\pi_{\mathbb{R}^N}$, we have $v_j \neq 0$ for all $j = 1, \ldots, N+M$ as well as $v_i \neq -v_j$ for all $i \neq j$. Further, $\pi_{\mathbb{R}^N}(x) = 0$ implies $\boldsymbol{\alpha} = 0$. This yields (IM1).

By assumption, we observe that

$$\pi_{\mathbb{R}^M}(\mathcal{L}) = \left\{ \sum_{j=1}^{N+M} n_j w_j : (n_1, \ldots, n_{N+M}) \in \mathbb{Z}^{N+M} \right\}$$

is dense in $\mathbb{R}^M$. This yields that we may find M indices $j_1, \ldots, j_M \in \{1, \ldots, N+M\}$ such that $\{w_{j_1}, \ldots, w_{j_M}\}$ is a linearly independent set, otherwise $\pi_{\mathbb{R}^M}(\mathcal{L})$ would not be an M-dimensional subgroup of $\mathbb{R}^M$.

Furthermore, there has to exist at least one additional index $j_0 \in \{1, \ldots, N+M\} \setminus \{j_1, \ldots, j_M\}$ with $w_{j_0} \neq 0$, otherwise $\pi_{\mathbb{R}^M}(\mathcal{L})$ would be isomorphic to $\mathbb{Z}^M$ which contradicts denseness.

Put $\mathcal{N} = \{w_{j_1}, \ldots, w_{j_M}\}$ and $\mathcal{M} = \{w_j \mid w_j \notin \mathcal{N}, w_j \neq 0\}$. By the preceding discussion, the set $\mathcal{S} = \mathcal{N} \cup \mathcal{M}$ then contains at least $M+1$ elements.

Now observe that each M-tuple of vectors out of $\mathcal{S}$ defines a regular matrix B which generates a lattice $\Gamma_B \leq \mathbb{R}^M$. Therefore each such M-tuple gives rise to an M-dimensional torus $\mathbb{T}_B = \mathbb{R}^M / \Gamma_B$. By $\pi_B : \mathbb{R}^M \to \mathbb{T}_B$ we denote the canonical projection. We assign to each $w \in \{w_1, \ldots, w_{N+M}\} \setminus \mathcal{N}$ a value $c(w) = \pi(w) \in \mathbb{T}_B$. Clearly, we have exactly N vectors $c(w)$ such that a least one of them satisfies $c(w) \neq 0$. Thus, we may label the $c(w)$ from 1 to N. We define a rotation

$$R : \mathbb{Z}^N \times \mathbb{T}_B \to \mathbb{T}_B : (n_1, \ldots, n_N, x) \mapsto \sum_{i=1}^{N} n_i c_i \mod \mathbb{T}_B.$$

Then $\mathcal{O}_R(x)$ is dense in $\mathbb{T}_B$ for all $x \in \mathbb{T}_B$ if and only if there is some c_i such that $B^{-1}(c_i)$ consists of rationally independent entries. Note that $\pi_{\mathbb{R}^M}(\mathcal{L}) \cap [0,1]^M$ is also dense in $[0,1]^M$ and $[0,1]^M$ may be identified in a natural way with the M-dimensional torus $\mathbb{T}^M$. Since $\mathbb{T}^M$

is homeomorphic to $\mathbb{T}_B$, denseness of $\pi_{\mathbb{R}^M}(\mathcal{L}) \cap [0,1]^M$ ensures minimality of R. However, this is only the case if some c_i is rationally independent with respect to B. Thus, we may choose vectors $w_{j_0} \in \mathcal{M}$ and $w_{j_1}, \ldots, w_{j_M} \in \mathcal{N}$ such that w_{j_0} is rationally independent with respect to $\begin{pmatrix} w_{j_1} & \ldots & w_{j_M} \end{pmatrix}^{-1}$. This shows (IM2). ☐

Remark 4.19. We want to point out that also the converse direction holds: every irrational matrix generates a lattice such that the canonical projections are injective and dense, respectively. However, we will not need this statement in our further discussions.

Remark 4.20. Summarizing the proof of the previous lemma, there exists a crucial connection between model sets arising from Euclidean CPS and irrational rotations on a topological torus.

To be more precise, let $A = \begin{pmatrix} u_1 & \ldots & u_{N+M} \end{pmatrix}$ be the generating matrix of a lattice $\mathcal{L}$ for a Euclidean CPS $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$. Then integer linear combinations of $v_i = \pi_{\mathbb{R}^N}(u_i)$, $i = 1, \ldots, N+M$, determine the possible points of the model set $\curlywedge(W)$. Let $w_i = \pi_{\mathbb{R}^M}(u_i)$, $i = 1, \ldots, N+M$. Then M vectors $w_{j_1}, \ldots, w_{j_M}$ span a lattice in $\mathbb{R}^M$ and thus determine a topological torus $\mathbb{T}$, while the remaining N vectors $w_{i_1} \ldots, w_{i_N}$ define a minimal $\mathbb{Z}^N$-action on $\mathbb{T}$. This action controls whether points of L are included in $\curlywedge(W)$ or not, i.e., if there exist $k_1, \ldots, k_N \in \mathbb{Z}$ such that $\sum_{i=1}^N k_i w_{j_i} \mod \mathbb{T} \in W$, then there exist $k_{N+1}, \ldots, k_{N+M} \in \mathbb{Z}$ such that $\sum_{i=1}^{N+M} k_i v_i \in \curlywedge(W)$.

4.3 Delone Dynamical Systems of FLC sets

Let G be a lca and second-countable group. Throughout this section we fix $r > 0$. We denote the set of all r-uniformly discrete subsets of G by

$$\mathcal{U} = \mathcal{U}_r(G) = \{\Gamma \subseteq G \mid d(g,h) \geq r \text{ for all distinct } g,h \in \Gamma\}.$$

There are different approaches to define a topology on $\mathcal{U}$. The first way is to define a uniform structure on $\mathcal{U}$ (compare [Kel75] for background on uniform structures). We will follow the depiction in [Sch99]. Let $K \subseteq G$ be compact and $V \subseteq G$ be an open neighbourhood of the identity. On $\mathcal{U}$ we then define the sets

$$U(K,V) = U_{LT}(K,V) = \{(\Gamma,\Gamma') \in \mathcal{U} \times \mathcal{U} \mid (v+\Gamma) \cap K = \Gamma' \cap K \text{ for some } v \in V\}.$$

Then the collection $\mathcal{C} = \{U(K,V) \mid K \text{ is compact and } V \text{ is an open neighbourhood of } 0\}$ defines the base of an uniform structure on $\mathcal{U}$. Hence, those entourages induce a basis for a topology on $\mathcal{U}$ in the following sense: a set $O \subseteq \mathcal{U}$ is open if and only if for all $\Gamma \in O$ there exists some $U(K,V) \in \mathcal{C}$ such that

$$U(K,V)[\Gamma] = \{\Gamma' \mid (\Gamma,\Gamma') \in U(K,V)\} \subseteq O.$$

The so-generated topology is called *local topology (LT)*. Note that this approach also works if G was not supposed to be second-countable.

For the second approach, we explicitly use that G is second-countable and thus metrizable. The latter implies that $\mathcal{U}$ is metrizable itself (compare [MR13]). We can define a distance on $\mathcal{U}$ given by

$$\mathrm{dist}_{\mathrm{LT}}(\Gamma,\Gamma') = \inf\{\varepsilon > 0 \mid \exists s \in B_\varepsilon(0) : (\Gamma - s) \cap B_{1/\varepsilon}(0) = \Gamma' \cap B_{1/\varepsilon}(0)\}.$$

Define furthermore

$$d(\Gamma,\Gamma') = d_{\mathrm{LT}}(\Gamma,\Gamma') = \min\left\{\frac{1}{\sqrt{2}}, \mathrm{dist}_{\mathrm{LT}}(\Gamma,\Gamma')\right\}.$$

This means that two points are close if they coincide on a large ball up to a small translation.

Lemma 4.21 ([LMS02])**.** *$(\mathcal{U}, d)$ is a metric space.*

Recall that the metric on G is proper, that is, $B_r(g)$ is relatively compact for all $r > 0$ and $g \in G$. Hence, we obtain

Lemma 4.22. *The topology on $\mathcal{U}$ induced by d is the local topology.*

Throughout this thesis, we will mostly work with the second approach. However, in both cases there is a canonical G-action on $\mathcal{U}$ given by

$$G \times \mathcal{U} \to \mathcal{U} : (g, \Gamma) \mapsto \Gamma - g.$$

Lemma 4.23 ([BLM07])**.** *The canonical G-action on $\mathcal{U}$ given as above is continuous.*

Suppose $\Lambda \in \mathcal{U}$ is an r-uniformly discrete set. Then we call

$$\Omega(\Lambda) = \Omega_{\mathrm{LT}}(\Lambda) = \mathrm{cl}_{\mathrm{LT}}\{\Lambda - g \mid g \in G\},$$

equipped with the natural G-action on $\mathcal{U}$, *(Delone) dynamical system* (or *(dynamical) hull*) of Λ. Sometimes the hull is also referred to as *(mathematical) quasicrystal*. In this context, we will sometimes denote the G-action on $\Omega(\Lambda)$ as φ. There is a crucial connection between geometrical properties of Λ and topological properties of its hull.

Lemma 4.24 ([Sch99])**.** *The following are equivalent:*

(i) $\Lambda \in \mathcal{U}$ *has FLC.*

(ii) $\Omega(\Lambda)$ *is compact.*

In either case, $(\Omega(\Lambda), d_{\mathrm{LT}})$ *is a complete metric space.*

There are also basic connections between geometrical properties of Λ and dynamical properties of $\Omega(\Lambda)$.

Proposition 4.25 ([Sch99])**.** *Assume $\Lambda \in \mathcal{U}$ has FLC. Then the following holds.*

(i) Λ *is repetitive if and only if* $(\Omega(\Lambda), G)$ *is minimal.*

(ii) Λ *has UPF if and only if* $(\Omega(\Lambda), G)$ *is uniquely ergodic.*

Proposition 4.26 ([Rob07])**.** *Assume $\Lambda \in \mathcal{U}$ has FLC. Then Λ is aperiodic if and only if $(\Omega(\Lambda), G)$ is free.*

In the context of Cut and Project Schemes we obtain as a direct consequence of Lemma 4.12 and Proposition 4.25 the following statement.

Proposition 4.27. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$.*

(i) If W is generic, then $(\Omega(\curlywedge(W)), G)$ is minimal.

(ii) If $\Theta_H(\partial W) = 0$, then $(\Omega(\curlywedge(W)), G)$ is uniquely ergodic.

Concluding this section we state a useful connection between dynamical hulls of Delone sets and the structure group of CPS.

Proposition 4.28. *Let $(G, H, \mathcal{L})$ be a CPS and $\Lambda \subseteq G$ a Delone set with FLC such that $\Lambda \subseteq L$. Then, for $\Gamma \in \Omega(\Lambda)$ the following assertions are equivalent:*

(i) $\Gamma \subseteq L$.

(ii) Γ *contains one point of L.*

Proof. $(i) \Rightarrow (ii)$: This is obvious.

$(ii) \Rightarrow (i)$: Let $x \in \Gamma \cap L$ be given. By FLC, there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ with $\Gamma_n = \Lambda + g_n \to \Gamma$, $n \to \infty$, and we may assume $x \in \Gamma_n$ for all $n \in \mathbb{N}$. We then have $x \in L$ as well as $x \in L + g_n$ and hence $g_n \in L$ for all $n \in \mathbb{N}$. Thus, we obtain $\Gamma_n \subseteq L$ for all $n \in \mathbb{N}$. Consider now an arbitrary point $y \in \Gamma$. As $\Gamma_n \to \Gamma$ and $x \in \Gamma_n \cap \Gamma$, we infer by FLC that $y \in \Gamma_n$ for sufficiently large n. This implies $y \in L$. ☐

4.4 The Torus Parametrization

Let $(G, H, \mathcal{L})$ be a CPS. Since $\mathcal{L} \subseteq G \times H$ is a lattice, the quotient $\mathbb{T} = (G \times H)/\mathcal{L}$ is a compact abelian group. A natural G-action on $\mathbb{T}$ (the so-called *Kronecker flow*) is given by

$$\omega : (u, [s,t]_{\mathcal{L}}) \mapsto [s + u, t]_{\mathcal{L}},$$

where $[s,t]_{\mathcal{L}}$ denotes the equivalence class of $(s,t) \in G \times H$. Then $(\mathbb{T}, G)$ is a minimal dynamical system (compare [Rob07]). Depending on the situation we will also refer to the G-action on $\mathbb{T}$ as ω.

Now consider a window $W \subseteq H$. We say the window is *irredundant* if it has no non-trivial translation invariance, that is, $\{h \in H \mid h + W = W\} = \{0\}$. Note that, if ∂W is irredundant, then W is irredundant, too. Under the assumption of irredundancy, it turns out that $(\mathbb{T}, G)$ is a factor of $(\Omega(\curlywedge(W)), G)$.

Proposition 4.29 ([BLM07])**.** *Let $(G, H, \mathcal{L})$ be a CPS with proper and irredundant window $W \subseteq H$ and $\Lambda \subseteq G$ such that*

$$\curlywedge(\mathrm{int}(W)) \subseteq \Lambda \subseteq \curlywedge(W).$$

Then there exists a unique factor map

$$\beta : (\Omega(\curlywedge(W)), G) \to (\mathbb{T}, G)$$

such that $\beta(\Lambda) = 0$ and

$$\beta(\Gamma) = [g, h]_{\mathcal{L}} \iff \curlywedge(\mathrm{int}(W) + h) - g \subseteq \Gamma \subseteq \curlywedge(W + h) - g \tag{4.4.1}$$

for $\Gamma \in \Omega(\Lambda)$.

The factor map β is also referred to as *torus parametrization* or *flow morphism*.

Before we discuss the properties of the torus parametrization, we want to point out a few important things regarding the definition of β.

Remark 4.30. Throughout this thesis we will often deal with $\Lambda = \curlywedge(W)$. In some situations (like in Chapter 5) we will also need to replace the window W by any of its translates $W + h$, $h \in H$. In this case, Proposition 4.29 yields a unique factor map

$$\beta_h : (\Omega(\curlywedge(W + h)), G) \to (\mathbb{T}, G)$$

which sends $\curlywedge(W + h)$ to 0. For $h = 0 \in H$ we will still write β instead of β_0.

Remark 4.31. The torus parametrization always exists as long as W is proper. If W is compact with $\mathrm{int}(W) = \emptyset$ then $\curlywedge(W)$ is a weak model set. In this case it is still possible to construct the dynamical hull of $\curlywedge(W)$, although then $\emptyset \in \Omega(\curlywedge(W))$. Thus, the empty set becomes a fixed point of the G-action on $\Omega(\curlywedge(W))$. On the other hand, there exists no fixed point of the G-action on $\mathbb{T}$ and hence there exists no semiconjugation. It is still possible to define such a mapping on $\Omega(\curlywedge(W)) \setminus \{\emptyset\}$ (compare [Sch99]).

Remark 4.32. Suppose $(G, H, \mathcal{L})$ is a CPS and $W \subseteq H$ is not irredundant. Let $H_W = \{h \in H \mid h + W = W\}$ denote the period group of W. Then it is always possible to construct a CPS $(G, H', \mathcal{L}')$ with an irredundant window $W' = W/H_W \subseteq H'$ such that for each $\Lambda \in \Omega(\curlywedge(W))$ with $\curlywedge(\mathrm{int}(W)) \subseteq \Lambda \subseteq \curlywedge(W)$ we have $\curlywedge(\mathrm{int}(W')) \subseteq \Lambda \subseteq \curlywedge(W')$. For a detailed discussion we refer to [LM06, Section 5] and [BLM07]. Thus, from now on, without loss of generality we may assume that all occurring windows in the context of CPS are irredundant.

Since the structure of the fibres of β will be crucial for later investigations, we want to discuss some basic properties of the fibres. The following lemma gives further insight in the structure of fibres of β. In fact, we can approximate elements of the fibres in a certain way. Note that similar arguments can also be found in [BLM07].

Lemma 4.33. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. For given $[0, h]_{\mathcal{L}} \in \mathbb{T}$, the following are equivalent:*

(i) $\Gamma \in \beta^{-1}([0, h]_{\mathcal{L}})$.

(ii) There exists a sequence $h_j \in L^*$ such that $\lim_{j\to\infty} h_j = h$ and

$$\lim_{j\to\infty} \curlywedge(W + h_j) = \Gamma.$$

Proof. $(i) \Rightarrow (ii)$: By $\beta(\Gamma) = [0, h]_{\mathcal{L}}$ we have

$$\Gamma \subseteq \curlywedge(W - h) \subseteq L.$$

By Proposition 4.28 we obtain a sequence $g_j \in L$ such that $\Lambda - g_j \to \Gamma$. Due to continuity of β, we then obtain

$$[0, h]_{\mathcal{L}} = \beta(\Gamma) = \lim_{j\to\infty} \beta\left(\varphi_{g_j}(\curlywedge(W))\right) = \lim_{j\to\infty} [0, g_j^*]_{\mathcal{L}}.$$

This easily implies convergence of $h_j = g_j^*$ to $h \in H$ for $j \to \infty$.

$(ii) \Rightarrow (i)$: This follows immediately from the continuity of β. □

Lemma 4.34 ([BLM07])**.** *Let $(G, H, \mathcal{L})$ be a CPS, $W \subseteq H$ a proper window and $\Lambda \subseteq G$ such that $\curlywedge(\mathrm{int}(W)) \subseteq \Lambda \subseteq \curlywedge(W)$. Then the following dichotomy holds.*

(i) If $(\partial W + h) \cap L^* = \emptyset$ then $[0, h]_{\mathcal{L}}$ has exactly one preimage under β.

(ii) If there exists $l \in L$ with $l^* \in \partial W + h$, then $\beta^{-1}([0, h]_{\mathcal{L}})$ contains at least two elements Γ and Γ' such that $l \in \Gamma$ and $l \notin \Gamma'$.

In particular, $[0, h]_{\mathcal{L}}$ has exactly one preimage under β if and only if $W + h$ is generic.

Proof. Clearly, $(W + h) \cap L^* = (\mathrm{int}(W) + h) \cap L^*$ if and only if $(\partial W + h) \cap L^* = \emptyset$. Consider first the case $(\mathrm{int}(W)+h)\cap L^* = (W+h)\cap L^*$. Then, by definition of β (compare Proposition 4.29), $[0, h]_{\mathcal{L}}$ has exactly one preimage under β.

Now let $l^* \in \partial W + h$ for some $l^* \in L^*$. Since L^* is dense in H and W is proper, we can find sequences $h_n = g_n^*$, $n \in \mathbb{N}$, and $h_n' = (g_n')^*$, $n \in \mathbb{N}$, with

- $\lim_{n\to\infty} h_n = \lim_{n\to\infty} h_n' = h$,
- $l^* \in \mathrm{int}(W) + h_n$ and $l^* \notin W + h_n'$ for all $n \in \mathbb{N}$.

By going over to subsequences if necessary, we may assume $(\varphi_{-g_n}(\Lambda))_{n\in\mathbb{N}} = (\Lambda + g_n)_{n\in\mathbb{N}}$ and $(\varphi_{-g_n'}(\Lambda))_{n\in\mathbb{N}} = (\Lambda + g_n')_{n\in\mathbb{N}}$ converge to Γ and Γ', respectively. Since

$$\varphi_{-g_n}(\Lambda) = \Lambda + g_n \supseteq \curlywedge(\mathrm{int}(W)) + g_n = \curlywedge(\mathrm{int}(W) + h_n) \ni l,$$

we obtain $l \in \Gamma$. In a similar way, we can show that at the same time $l \notin \Gamma'$. Hence, we obtain $\Gamma \neq \Gamma'$. As β is a G-map, we have

$$\begin{aligned}\beta(\Gamma) = \beta\left(\lim_{n\to\infty} \varphi_{-g_n}(\Lambda)\right) &= \lim_{n\to\infty} \beta(\varphi_{-g_n}(\Lambda)) = \lim_{n\to\infty} \omega_{-g_n}(\beta(\Lambda)) \\ &= \lim_{n\to\infty} \omega_{-g_n}(0) = \lim_{n\to\infty} [-g_n, 0]_{\mathcal{L}} = \lim_{n\to\infty} [0, h_n]_{\mathcal{L}} = [0, h]_{\mathcal{L}}.\end{aligned}$$

The same holds for Γ' and hence $\Gamma, \Gamma' \in \beta^{-1}([0, h]_{\mathcal{L}})$. □

As a direct consequence of Lemma 4.34 and Lemma 4.11 we obtain

Corollary 4.35. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. Then there exists some $h \in H$ such that $\sharp\beta^{-1}([0, h]_{\mathcal{L}}) = 1$.*

From the fact that L^* is countable and the above Lemma 4.34, we also immediately obtain that regularity of the window W has strong implications for the fibre structure.

Corollary 4.36 ([BLM07]). *Let $(G, H, \mathcal{L})$ be a CPS, $W \subseteq H$ a proper window and $\Lambda \subseteq G$ such that $\curlywedge(\mathrm{int}(W)) \subseteq \Lambda \subseteq \curlywedge(W)$. Further, assume that W is regular, i.e. $\Theta_H(\partial W) = 0$. Then for $\Theta_{\mathbb{T}}$-almost all $\xi \in \mathbb{T}$ the preimage $\beta^{-1}(\xi)$ is a singleton. In particular, the flow $(\Omega(\Lambda), G)$ is uniquely ergodic and measure-theoretically isomorphic to $(\mathbb{T}, G)$.*

In the following, we apply the methods introduced in Section 3.3 about (ir)regular extensions of equicontinuous dynamical systems to the setting of CPS. A first observation is the following.

Lemma 4.37 ([KR16]). *Suppose $(G, H, \mathcal{L})$ is a CPS and $W \subseteq H$ is a proper window. Then $(\mathbb{T}, G)$ is the MEF of $(\Omega(\curlywedge(W)), G)$.*

Now let $h \in H$ such that $(\partial W + h) \cap L^* = \emptyset$. Then, by Lemma 4.34(i), we have $\curlywedge(\mathrm{int}(W) + h) - g = \curlywedge(W + h) - g$. Thus, $\beta^{-1}([g, h]_{\mathcal{L}})$ is a singleton for every $g \in G$. Put

$$\mathcal{G}_W = \{h \in H \mid \sharp\beta^{-1}([g, h]_{\mathcal{L}}) = 1 \text{ for all } g \in G\}.$$

Comparing the proof of Lemma 4.11, it is not hard to see that $\mathcal{G}_W$ is a residual set. By definition of $\mathcal{G}_W$, Lemma 4.37 and the discussion in Section 3.3 we obtain

Lemma 4.38 ([Rob07]). *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$ and let $[g, h]_{\mathcal{L}} \in \mathbb{T}$. If $h \in \mathcal{G}_W$, then*

$$(\Omega(\curlywedge(W + h) - g), G)$$

is an almost automorphic system.

As a direct conclusion we may point out the following connection between (ir)regularity of model sets and (ir)regularity in the context of (ir)regular extensions.

Lemma 4.39. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. If $\curlywedge(W)$ is an (ir)regular model set, then $(\Omega(\curlywedge(W)), G)$ is an (ir)regular extension of $(\mathbb{T}, G)$.*

Regarding the dynamical invariant of entropy we obtain the following fundamental observation.

Lemma 4.40. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. If $\Theta_H(\partial W) = 0$ then $h_{\mathrm{top}}(\Omega(\curlywedge(W)), G) = 0$.*

Proof. The proof makes use of the concept of metric entropy. Since we won't need this concept elsewhere, we refer to [Wal82] for an introduction to this topic. By Corollary 4.36 the dynamical systems $(\Omega(\curlywedge(W)), G)$ and $(\mathbb{T}, G)$ are measure-theoretically isomorphic. Thus, the metric entropies of both systems have to agree. Since $(\mathbb{T}, G)$ is uniquely ergodic, its topological entropy agrees with its metric entropy. Since $h_{\mathrm{top}}(\mathbb{T}, G) = 0$ we obtain $h_{\mathrm{top}}(\Omega(\curlywedge(W)), G) = 0$. ☐

It is even possible to give an upper bound for the entropy in terms of the measure of the windows boundary.

Lemma 4.41 ([HR15]). *Let $(G, H, \mathcal{L})$ be a CPS with compact window $W \subseteq H$. Then we have*

$$h_{\mathrm{top}}(\Omega(\curlywedge(W)), G) \leq \frac{\log 2}{\mu(\mathcal{L})} \Theta_H(\partial W),$$

where $\mu(\mathcal{L})$ denotes the measure of a fundamental domain of $(R^N \times H)/\mathcal{L}$.

Remark 4.42. (i) Note that this estimation also holds in case of weak model sets.

(ii) This estimation yields another proof of Lemma 4.40 which omits the usage of factor maps.

(iii) The proof of the statement above makes use of the notion of *pattern entropy*, which is, roughly spoken, a measure for the complexity of a uniformly discrete set in terms of the number of different occurring patches with respect to some fixed van Hove sequence. For deeper discussion we refer to [HR15].

4.5 Geometrical properties and Delone Dynamical Systems of non-FLC sets

As seen in the definitions of Section 4.1 and Lemma 4.24, being FLC is a crucial requirement for point sets to define geometrical properties like repetitivity or to obtain compact dynamical systems associated to those sets. Suppose $\Gamma, \Gamma' \in \mathcal{U}$ are non-FLC sets. In general those two sets won't coincide on a large ball up to a small translation, which renders the local topology useless. We want to define a metric on $\mathcal{U}$ in which Γ and Γ' are close if they coincide on a large ball up to small translations of *all single points* in Γ and Γ'. It turns out that the dynamical hull of a Delone set with respect to this metric becomes compact. Furthermore, this approach gives us a method to define repetitivity for non-FLC sets.

Let G be a lca and second-countable group. By [MR13], the set $\mathcal{U}$ then becomes metrizable. For $\Gamma, \Gamma' \in \mathcal{U}$ let

$$\mathrm{dist}_{\mathrm{LRT}}(\Gamma, \Gamma') = \inf \left\{ \varepsilon > 0 \mid \Gamma \cap B_{1/\varepsilon}(0) \subseteq (\Gamma')_\varepsilon \text{ and } \Gamma' \cap B_{1/\varepsilon}(0) \subseteq (\Gamma)_\varepsilon \right\},$$

where $(\Gamma)_\varepsilon = \bigcup_{p\in\Gamma} B_\varepsilon(p)$. Define

$$d_{\mathrm{LRT}}(\Gamma, \Gamma') = \min \left\{ \frac{1}{\sqrt{2}}, \mathrm{dist}_{\mathrm{LRT}}(\Gamma, \Gamma') \right\}.$$

For $\Lambda \in \mathcal{U}$ we then define the corresponding *(Delone) dynamical system* (or *(dynamical) hull*)

$$\Omega(\Lambda) = \Omega_{\mathrm{LRT}}(\Lambda) = \mathrm{cl}_{\mathrm{LRT}}\{\Lambda - g \mid g \in G\}.$$

We call the topology induced by d_{LRT} *local rubber topology*. If no confusion arises, we omit mentioning the indices LT or LRT. The name "local rubber topology" is justified by the following lemma.

Lemma 4.43 ([MR13]). *(i) The space $(\mathcal{U}, d_{\mathrm{LRT}})$ is a compact Hausdorff space.*

(ii) For any $\Lambda \in \mathcal{U}$ the space $\Omega_{\mathrm{LRT}}(\Lambda)$ is compact. In particular, $(\Omega_{\mathrm{LRT}}(\Lambda), d_{\mathrm{LRT}})$ is a complete metric space.

Furthermore, the local rubber topology and the local topology coincide for FLC sets.

Corollary 4.44 ([FR14]). *Suppose $\Lambda \in \mathcal{U}$ has FLC. Then $\Omega_{\mathrm{LT}}(\Lambda) = \Omega_{\mathrm{LRT}}(\Lambda)$.*

On the geometrical side, we are now able to define a generalization of repetitivity for non-FLC sets. Let $\Gamma, \Gamma' \in \mathcal{U}$ and $R > 0$. Define

$$d_R(\Gamma, \Gamma') = \inf\{\varepsilon > 0 \mid \Gamma \cap B_R(0) \subseteq (\Gamma')_\varepsilon \text{ and } \Gamma' \cap B_R(0) \subseteq (\Gamma)_\varepsilon\}.$$

Note that this distance equals the Hausdorff distance between Γ and Γ' if both sets have no points outside of $B_R(0)$. We say a Delone set $\Gamma \in \mathcal{U}$ is *almost repetitive* (or *LRT-repetitive*) if for all $\delta > 0$ and $R > 0$ the set

$$\{l \in \Gamma \mid d_R(\Gamma, \Gamma - l) < \delta\}.$$

Lemma 4.45 ([FR14]). *Let $\Lambda \in \mathcal{U}$. Then Λ is almost repetitive if and only if $(\Omega_{\mathrm{LRT}}(\Lambda), G)$ is minimal.*

Part II

Irregular Cut and Project Schemes

Chapter 5

Irregular Cut and Project Schemes with Positive Entropy

In this chapter we are going to provide two classes of examples of Euclidean CPS $(\mathbb{R}^N, H, \mathcal{L})$ with irregular and proper window $W \subseteq H$ such that the corresponding Delone dynamical system yields positive topological entropy. We will introduce a simple criterion for positive entropy of the hull and relate this to the local structure of the window. As it turns out, this method will also apply to the case of weak model sets.

Throughout this section we assume that all occurring uniformly discrete sets have FLC unless mentioned otherwise. Further, by $(A_n)_{n\in\mathbb{N}}$ we denote the sequence of cubes of side-length $2n$ and volume $(2n)^N$ in $\mathbb{R}^N$ (compare Section 4.2). Note that this sequence is a tempered van Hove sequence.

5.1 Embedded Fullshifts and Topological Entropy

The key concept in providing positive topological entropy is that of embedded fullshifts. Assume $\Lambda \subseteq \mathbb{R}^N$ is a uniformly discrete set. An *embedded fullshift* in $\Omega(\Lambda)$ is a pair (Ξ, S) consisting of a closed subset $\Xi \subseteq \Omega(\Lambda)$ and a subset $S \subseteq \mathbb{R}^N$ such that the following holds:

(FS1) The set S has positive asymptotic density (recall Section 4.2) , i.e.

$$\nu_S = \limsup_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sharp(S \cap A_n) > 0.$$

(FS2) The set

$$U = \bigcup_{\Gamma\in\Xi} \Gamma$$

is uniformly discrete.

(FS3) For any subset $S' \subseteq S$ there exists a $\Gamma \in \Xi$ such that

$$\Gamma \cap S = S'.$$

The elements of S above are called *free points* of the embedded fullshift. The set U is called *grid* of the embedded fullshift. The quantity ν_S is the *asymptotic density* of the embedded fullshift.

If (Ξ, S) is an embedded fullshift in $\Omega(\Lambda)$ with $\Xi \subseteq \Omega'$ for some $\Omega' \subseteq \Omega(\Lambda)$ we say that Ω' *contains an embedded fullshift*.

We want to point out some properties of embedded fullshifts which follow immediately from the definition.

Lemma 5.1. *Let $\Lambda \subseteq \mathbb{R}^N$ be uniformly discrete with FLC. Assume (Ξ, S) is an embedded fullshift of $\Omega(\Lambda)$. Then the following holds.*

(i) We have $S \subseteq U$. In particular, S is uniformly discrete.

(ii) Let $\Xi' = \mathrm{cl}(\{\Gamma \in \Xi \mid \Gamma \cap S \neq \emptyset\})$. Then (Ξ', S) is also an embedded fullshift with $\Xi' \subseteq \Xi$.

(iii) For all $s \in \mathbb{R}^N$ the pair $(\varphi_s(\Xi), \varphi_s(S))$ is an embedded fullshift.

In the context of Cut and Project Schemes we obtain the following.

Lemma 5.2. *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. Let $\xi = [s, h]_{\mathcal{L}} \in \mathbb{T}$. If $\beta^{-1}(\xi)$ contains an embedded fullshift (Ξ, S) then $(\beta^{-1}(\xi), S)$ is an embedded fullshift in $\Omega(\curlywedge(W))$ with $U = \curlywedge(W + h) - s$.*

Proof. Let $\xi = [s, h]_{\mathcal{L}} \in \mathbb{T}$. First note, that by Proposition 4.29 all elements of $\beta^{-1}(\xi)$ are contained in $\curlywedge(W + h) - s$. Hence, for any subset $\Xi \subseteq \beta^{-1}(\xi)$ we then obtain uniform discreteness of $\bigcup_{\Gamma \in \Xi} \Gamma$. Thus, (FS2) holds. By Proposition 4.29 and Lemma 4.34 we then infer $U = \curlywedge(W + h) - s$. It is obvious that (FS1) and (FS3) are fulfilled for $(\beta^{-1}(\xi), S)$. ☐

Remark 5.3. The points of S are free in the sense that we can choose any subset of S and exactly this will be the subset from S appearing in some $\Gamma \in \Xi$. In later arguments we will not only have to control occurrence of points of S but also non-occurrence of points of S. We will need the set U in order to treat this non-occurrence.

In the following we provide a simple characterization for existence of an embedded fullshift.

Proposition 5.4. *Let $\Lambda \subseteq \mathbb{R}^N$ be a uniformly discrete set. Then $\Omega(\Lambda)$ contains an embedded fullshift if and only if there exists $S \subseteq \mathbb{R}^N$ and a uniformly discrete set $U \subseteq \mathbb{R}^N$ with the following properties.*

(i) The set S has positive asymptotic density.

(ii) For all finite $F \subseteq S$ and $a \in \{0,1\}^F$, there exists a $\Gamma \in \Omega(\Lambda)$ with $\Gamma \subseteq U$ such that for $s \in F$ holds

$$s \in \Gamma \iff a_s = 1.$$

Proof. If $\Omega(\Lambda)$ contains an embedded fullshift there clearly exist $S \subseteq \mathbb{R}^N$ and a uniformly discrete $U \subseteq \mathbb{R}^N$ satisfying (i) and (ii). Conversely, if there exist $S \subseteq \mathbb{R}^N$ and a uniformly discrete $U \subseteq \mathbb{R}^N$ satisfying (i) and (ii) we may define

$$\Xi' = \{\Gamma \in \Omega(\Lambda) \mid \Gamma \cap S \neq \emptyset \text{ and } \Gamma \subseteq U\}.$$

Let $\Xi = \mathrm{cl}(\Xi')$. Then, all elements in Ξ are contained in U and a simple compactness argument shows that for any subset $S' \subseteq S$ there exists a $\Gamma \in \Xi$ with $\Gamma \cap S = S'$. Hence, (Ξ, S) is an embedded fullshift contained in $\Omega(\Lambda)$. ☐

Corollary 5.5. *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with compact window $W \subseteq H$. Assume $\Omega(\curlywedge(W))$ satisfies conditions (i) and (ii) of Proposition 5.4. Let $\Xi' = \{\Gamma \in \Omega(\curlywedge(W)) \mid \Gamma \cap S \neq \emptyset$ and $\Gamma \subseteq U\}$ and $U' = \bigcup_{\Gamma \in \Xi'} \Gamma$. Then for all $s \in S$ we have*

$$S \subseteq U' \subseteq t + L.$$

Proof. Shifting S and U by $-s$ for $s \in S$, we may assume without loss of generality that $0 \in S$ and $s = 0$. Since $0 \in L$ we may infer from Proposition 4.28 that any $\Gamma \in \Xi = \mathrm{cl}(\Xi')$ containing 0 must be contained in L. This yields

$$\tilde{U} = \bigcup_{\Gamma \in \Xi, 0 \in \Gamma} \Gamma \subseteq L.$$

Since $\tilde{U} \subseteq U$ is a discrete set, a compactness argument shows $\tilde{U} = U'$. Hence, $U' \subseteq L$ and since any $s \in S$ is contained in U', the statement holds. ☐

Remark 5.6. Note that this corollary allows $\mathrm{int}(W) = \emptyset$. Hence, it also holds in the setting of weak model sets.

The relevance of embedded fullshifts as a sufficient condition for positive topological entropy comes from the following lemma.

Lemma 5.7. *Let $\Lambda \subseteq \mathbb{R}^N$ be a uniformly discrete set with FLC. If $\Omega(\Lambda)$ contains an embedded fullshift of asymptotic density ν_S, then*

$$h_{\mathrm{top}}(\varphi) \geq \nu_S \cdot \log 2.$$

Proof. Let S be the set of free points and U the grid of the embedded fullshift. By (FS2) we may assume U is r-uniformly discrete for some $r > 0$. Consider $\Gamma, \Gamma' \in \Omega(\Lambda)$ with $\Gamma, \Gamma' \subseteq U$ and $s \in \Gamma$ and $s \notin \Gamma'$ for some $s \in S$. By uniform discreteness of U the set Γ' then does not contain a point in $B_r(s)$. This yields

$$d(\varphi_s(\Gamma), \varphi_s(\Gamma')) \geq r.$$

Hence, any pair $\Gamma, \Gamma' \in \Omega(\Lambda)$ which satisfies the above for some $s \in S \cap A_n$ is (r, n)-separated. Consider now an arbitrary $\nu < \nu_S$. Then there exists an arbitrarily large n with

$$\sharp(S \cap A_n) \geq \nu \cdot \mathrm{Leb}(A_n).$$

By the assumption on existence of an embedded fullshift, for each finite subset F of $S \cap A_n$ we can choose an element $\Gamma_F \in \Omega(\Lambda)$ with $\Gamma_F \cap S = F$. Then all elements Γ_F are (r, n)-separated by the considerations at the beginning of the proof. Hence, we have

$$N^{\Omega(\Lambda)}(\varphi, r, n) \geq 2^{\nu \cdot \mathrm{Leb}(A_n)}.$$

But this implies

$$h_r^{\Omega(\Lambda)}(\varphi) \geq \nu \cdot \log 2.$$

As $\nu < \nu_S$ was arbitrary we infer $h_r^{\Omega(\Lambda)}(\varphi) \geq \nu_S \cdot \log 2$. Now the desired statement follows from $h_{\mathrm{top}}(\varphi) \geq h^{\Omega(\Lambda)}(\varphi)$. ☐

5.2 Independence of sets

In the context of Euclidean Cut and Project Schemes, we want to provide a condition for the existence of an embedded fullshift. Thus, let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS. In order to provide also a criterion for the existence of embedded fullshifts in the case of hulls arising from weak model sets we only assume $W \subseteq H$ to be compact.

The problem of finding an embedded fullshift with set of free points $S \subseteq L$ in the associated dynamical system is actually related to analysing the local structure of the window W in some neighbourhood of the points s^* for $s \in S$. In order to get a first idea on this issue the following observations may be helpful.

Let $F \subseteq L$ be a finite set, $a \in \{0, 1\}^F$ arbitrary and $h \in H$ be given. Now assume that

$$\left(\bigcap_{s \in F : a_s = 1} (W - s^*) \setminus \bigcup_{s \in F : a_s = 0} (W - s^*) \right) \cap (L^* - h) \neq \emptyset.$$

Then there exists an $l \in L$ satisfying for any $s \in F$ that

$$l^* - h \in W - s^* \iff a_s = 1.$$

For $s \in F$ this yields

$$s \in \curlywedge(W + h) - l \iff a_s = 1.$$

Thus, the set $\curlywedge(W + h) - l$ respects the choice of $F \subseteq L$ given by a. Our dealing below will build on this observation. However, two additional points will come up:

- We have to simultaneously deal with all finite subsets F of a subset $S \subseteq L$. In order to still provide the uniform discrete subset U necessary for an embedded fullshift, we will need to require that the set $S^* = \{s^* \mid s \in S\}$ is relatively compact (see Lemmas 5.8 and 5.13).
- We will allow for one overall shift by $\theta \in H$.

Motivated by the preceding considerations we give the following definition. The finite index set F appearing in the definition will later be a subset of L (or L^*, respectively).

Let $D \subseteq H$ be given. A finite family C_s, $s \in F$, of subsets of H is *independent with respect to D* if for all $a \in \{0,1\}^F$ we have

$$\left(\bigcap_{s\in F: a_s=1} C_s \setminus \bigcup_{s\in F: a_s=0} C_s \right) \cap D \neq \emptyset.$$

An infinite family of sets is called *independent with respect to D* if the condition above holds for each finite subfamily. We say the window W is *independent in $P \subseteq L^*$ with respect to D*, if the family $W - p$, $p \in P$, is independent with respect to D.

The following lemma relates these concepts to the existence of embedded fullshifts. This lemma is our main tool to construct embedded fullshifts (and hence, by Lemma 5.7, examples with positive topological entropy). In fact, we will apply the lemma in two situations, namely for proper windows and for windows with empty interior but of positive measure. The lemma is formulated in a general version that includes two parameters $h, \theta \in H$.

Lemma 5.8 (Basic criterion for embedded fullshifts)**.** *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with compact window $W \subseteq H$ and $h, \theta \in H$. If $\curlywedge(W+\theta)$ possesses a subset S of positive asymptotic density such that $S^* = \{s^* \mid s \in S\}$ is relatively compact and $W + \theta$ is independent in S^* with respect to $L^* + (\theta - h)$, then $\Omega(\curlywedge(W+h))$ contains an embedded fullshift.*

Proof. We will show that the conditions (i) and (ii) of Proposition 5.4 are met for S as in the statement of this lemma and

$$U = \curlywedge(W + \theta - S^*).$$

Note that U is indeed uniformly discrete as $W + \theta - S^*$ is compact.

Condition (i) is met by assumption. To show condition (ii) fix a finite subfamily $F \subseteq S$ and let $a \in \{0,1\}^F$. Then, by independence of $W + \theta$ in S^* with respect to $L^* + (\theta - h)$, we have

$$\left(\bigcap_{s\in F: a_s=1} (W + \theta - s^*) \setminus \bigcup_{s\in F: a_s=0} (W + \theta - s^*) \right) \cap (L^* + \theta - h) \neq \emptyset.$$

Thus there exists an

$$m^* \in L^* + \theta - h$$

such that

$$m^* \in W + \theta - s^* \iff a_s = 1 \tag{5.2.1}$$

for all $s \in F$. Further, by the symmetry $L^* = -L^*$ we have

$$m^* = \theta - h - k^* \tag{5.2.2}$$

for some $k \in L$. Combining this with (5.2.1) we obtain

$$s \in \curlywedge(W + h) + k \text{ if and only if } a_s = 1. \tag{5.2.3}$$

Moreover, we have

$$\Gamma = \curlywedge(W + h) + k = \curlywedge(W + h + k^*) = \curlywedge(W + \theta - m^*) \subseteq U,$$

where the last equality holds due to (5.2.2) and the inclusion uses that m^* belongs to $W + \theta - S^*$ by (5.2.1). Thus, Γ belongs to $\Omega(\curlywedge(W+h))$ with $\Gamma \subseteq U$, and due to (5.2.3) we have

$$s \in \Gamma \text{ if and only if } a_s = 1.$$

This finishes the proof. $\square$

A slightly more specific notion of independence is given in the following definition. It will be needed in particular to obtain further information about embedded fullshifts in the case of proper model sets.

Let $D \subseteq H$ be given. A finite family C_s, $s \in F$, of subsets of H is *locally independent in* $0 \in H$ *with respect to* D if for all $a \in \{0,1\}^F$ we have

$$0 \in \mathrm{cl}\left(\left(\bigcap_{s\in F: a_s = 1} C_s \setminus \bigcup_{s \in F: a_s = 0} C_s\right) \cap D\right).$$

An infinite family of sets is called *locally independent in* $0 \in H$ *with respect to* D if the above condition holds for each finite subfamily. We say the window W is *locally independent in* $P \subseteq L^*$ *with respect to* D, if the family $W - p$, $p \in P$, is locally independent in $0 \in H$ with respect to D.

It is easy to see that the following characterization holds.

Corollary 5.9. *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. Let $D \subseteq H$ and $P \subseteq L^*$. Then W is locally independent in P with respect to D if and only if*

$$0 \in \mathrm{cl}\left(\left(\bigcap_{s\in F: a_s = 1} (W - s^*) \setminus \bigcup_{s \in F: a_s = 0} (W - s^*)\right) \cap D\right)$$

for any finite $F \subseteq P$ and any $a \in \{0,1\}^F$.

We obtain the following connection between embedded fullshifts in fibres of the factor map and local independence of proper windows (recall Remark 4.30 for a discussion of the notations used in the following).

Corollary 5.10. *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS, $W \subseteq H$ proper and $h, \theta \in H$. Assume that $\curlywedge(W+\theta)$ possesses a subset S of positive asymptotic density such that $W + \theta$ is locally independent in $S^* = \{s^* \mid s \in S\}$ with respect to $L^* + (\theta - h)$. Then there is an embedded fullshift contained in $\beta_h^{-1}([0, \theta - h]_{\mathcal{L}})$.*

Proof. This follows by extending the proof of the previous Lemma 5.8. Fix a finite subfamily $F \subseteq S$ and $a \in \{0,1\}^F$. By Corollary 5.9 we can choose some $m^* \in L^* + \theta - h$ such that

- $m^* \in W + \theta - s^* \iff a_s = 1$,
- m^* is arbitrarily close to 0.

This means that we can find a sequence $m_j^* \in L^* + \theta - h$, $j \in \mathbb{N}$, such that

- $m_j^* \in W + \theta - s^* \iff a_s = 1$ for all $j \in \mathbb{N}$,
- $\lim_{j\to\infty} m_j^* = 0$.

Let $V \subseteq H$ be a compact neighbourhood of 0. Without loss of generality we may assume that $m_j^* \in V$ for all $j \in \mathbb{N}$.

If now $k_j \in L$ are chosen such that $m_j^* = \theta - h - k_j^*$ (compare Equation (5.2.2)) then we obtain

$$s \in \Gamma_j = \curlywedge(W+h) + k_j \text{ if and only if } a_s = 1. \tag{5.2.4}$$

Moreover, we have

$$\Gamma_j = \curlywedge(W + h + k_j^*) = \curlywedge(W + \theta - m_j^*) \subseteq \curlywedge(W + \theta - V) = U,$$

where U is uniformly discrete since $W + \theta - V$ is compact.

Now, $(\Gamma_j)_{j\in\mathbb{N}}$ is a sequence in the compact space $\Omega(\curlywedge(W+h))$. Hence, it possesses an accumulation point $\Gamma \in \Omega(\curlywedge(W+h))$. As Γ_j is a subset of U and U is uniformly discrete, $\lim_{j\to\infty}\Gamma_j = \Gamma$ and Equation (5.2.4) yield

$$s \in \Gamma \text{ if and only if } a_s = 1.$$

As W is proper, β_h is continuous. This gives

$$\begin{aligned}\beta_h(\Gamma) &= \lim_{j\to\infty}\beta_h(\curlywedge(W+h)+k_j) = \lim_{j\to\infty}\beta_h(\curlywedge(W+h+k_j^*))\\ &= \lim_{j\to\infty}[0,k_j^*]_{\mathcal{L}} = \lim_{j\to\infty}[0,\theta-h-m_j]_{\mathcal{L}} = [0,\theta-h]_{\mathcal{L}}.\end{aligned}$$

Thus, the Γ constructed above are all contained in the fibre $\beta_h^{-1}([0,\theta-h]_{\mathcal{L}})$. Hence we obtained an embedded fullshift in that fibre. □

Now we will provide a sufficient condition for applicability of Corollary 5.10. Our constructions of proper windows which generate hulls with positive topological entropy will mainly be based on this condition. Thus, we assume in the following that $W \subseteq H$ is a proper window.

We say a finite family of sets C_s, $s \in F$, of subsets of H is *locally topologically independent in* $0 \in H$ if for all $a \in \{0,1\}^F$ we have

$$0 \in \mathrm{cl}\left(\mathrm{int}\left(\bigcap_{s\in F: a_s=1} C_s \setminus \bigcup_{s\in F: a_s=0} C_s\right)\right).$$

An infinite family of sets is called *locally topologically independent in* $0 \in H$ if the condition above holds for each finite subfamily. We say the window W is *locally topologically independent in* $P \subseteq L^*$ if the family $W - p$, $p \in P$, is locally topologically independent in 0.

Lemma 5.11. *Any family of subsets of H which is locally topologically independent in $0 \in H$ is locally independent in 0 with respect to any dense subset $D \subseteq H$.*

Proof. Consider an arbitrary finite subfamily C_s, $s \in F$, of the original family and let $a \in \{0,1\}^F$ be given. We define

$$C(a) = \bigcap_{s\in F: a_s=1} C_s \setminus \bigcup_{s\in F: a_s=0} C_s.$$

By assumption we have $0 \in \mathrm{cl}(\mathrm{int}(C(a)))$. Since D is dense in H, the intersection $\mathrm{int}(C(a)) \cap D$ is dense in $\mathrm{int}(C(a))$. Thus, we may choose a sequence $(\theta_j)_{j\in\mathbb{N}}$ in $\mathrm{int}(C(a)) \cap D$ such that $\lim_{j\to\infty}\theta_j = 0$. □

Lemma 5.12 (Topological criterion for embedded fullshifts)**.** *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$ and suppose $\theta \in H$. Assume that there exists a subset $S \subseteq \curlywedge(W+\theta)$ of positive asymptotic density such that $W+\theta$ is locally topologically independent in $S^* = \{s^* \mid s \in S\}$. Then the fibre $\beta_h^{-1}([0,\theta-h]_{\mathcal{L}})$ contains an embedded fullshift for every $h \in H$.*

Proof. As $W+\theta$ is locally topologically independent in S^* and $L^* + (\theta - h)$ is dense in H for all $h \in H$, the preceding Lemma 5.11 gives that $W+\theta$ is locally independent in S^* with respect to $L^* + (\theta-h)$ for all $h \in H$. As W is proper, we can now apply Corollary 5.10 to obtain that the fibre $\beta_h^{-1}([0,\theta-h]_{\mathcal{L}})$ contains an embedded fullshift. □

Concluding this section we want to adapt the concepts above to the case of weak models sets. Since we have $\mathrm{int}(W) = \emptyset$ in this case, we need to replace open sets by sets of positive measure and invoke uniform distribution in order to prove analogous statements. As a result we will obtain a criterion for embedded fullshifts for compact $W \subseteq H$ with positive measure.

We say a finite family of sets C_s, $s \in F$, of subsets of H is *metrically independent* if for all $a \in \{0,1\}^F$ we have

$$\Theta_H\left(\bigcap_{s\in F:a_s=1} C_s \setminus \bigcup_{s\in F:a_s=0} C_s\right) > 0.$$

An infinite family of subsets of H is called *metrically independent* if the condition above holds for each finite subfamily. Further, we say the window W is *metrically independent in* $P \subseteq L^*$ if the family $W - p$, $p \in P$, is locally metrically independent.

Lemma 5.13 (Metric criterion for embedded fullshifts). *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with compact window $W \subseteq H$ and let $\theta \in H$. Assume that there exists a subset $S \subseteq ⅄(W+\theta)$ of positive asymptotic density such that $S^* = \{s^* \mid s \in S\}$ is compact and $W+\theta$ is metrically independent in S^*. Then $\Omega(⅄(W+h))$ contained an embedded fullshift for Θ_H-almost every $h \in H$.*

Proof. Let F be a finite subset of S and let $a \in \{0,1\}^F$ be given. Consider the family $W+\theta-s^*$, $s \in F$, and define

$$\mathcal{W}(a) = \bigcap_{s\in F:a_s=1} W+\theta-s^* \setminus \bigcup_{s\in F:a_s=0} W+\theta-s^*.$$

Since $W+\theta$ is metrically independent in S^* we have

$$\Theta_H(\mathcal{W}(a)) > 0.$$

By Theorem 4.13 we thus obtain that the density of

$$⅄(\mathcal{W}(a) - \theta + h)$$

is positive for almost every $h \in H$. By excluding a set of measure zero, we therefore obtain a set $\mathcal{H}(a) \subseteq H$ of full measure such that for every $h \in \mathcal{H}(a)$ we have

$$(L^* + \theta - h) \cap \mathcal{W}(a) \neq \emptyset.$$

Intersecting over the countable family of all finite $F \subseteq S$ and $a \in \{0,1\}^F$ we obtain a set $\mathcal{H} \subseteq H$ of full measure. Then for each $h \in \mathcal{H}$ we have

$$(L^* + \theta - h) \cap \mathcal{W}(a) \neq \emptyset$$

for arbitrary $F \subseteq S$ and $a \in \{0,1\}^F$. Hence, $W+\theta$ is locally independent in S^* with respect to $L^*+\theta-h$ for each $h \in \mathcal{H}$. Given this, Lemma 5.8 implies the assertion. ☐

5.3 Embedded Fullshifts and Unique Ergodicity

In this section we study how existence of embedded fullshifts of sufficiently high density prevents unique ergodicity of the corresponding hull. Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS and $W \subseteq H$ be a compact window. Recall that, if the associated system $(\Omega(⅄(W)), \mathbb{R}^N)$ is uniquely ergodic, by Proposition 4.25, the density (compare Section 4.2)

$$\nu_\Gamma = \lim_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sharp(\Gamma \cap A_n)$$

exists for every $\Gamma \in \Omega(⅄(W))$ and is independent of Γ (as this is just the patch frequency of the patch $(\{0\}, r/2)$, where r is the minimal distance between points in Γ). Based on this basic observation we can now show that $(\Omega(⅄(W)), \mathbb{R}^N)$ cannot be uniquely ergodic if it contains an embedded fullshift with set of free points S and grid U such that ν_S is large compared to ν_U.

Proposition 5.14. *Let $(\mathbb{R}^N, H, \mathcal{L})$ be a CPS with compact window $W \subseteq H$. Suppose that $\Omega(\curlywedge(W))$ contains an embedded fullshift with set of free points S and grid U such that*

$$\nu_S > \frac{\nu_U}{2}.$$

Then $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ is not uniquely ergodic. This applies in particular if W is proper and $\Omega(\curlywedge(W))$ contains a fullshift embedded in a fibre with asymptotic density

$$\nu_S > \frac{\Theta_H(W)}{2\mu(\mathcal{L})},$$

where $\mu(\mathcal{L})$ denotes the measure of a fundamental domain of $(\mathbb{R}^N \times H)/\mathcal{L}$.

Proof. Let (Ξ, S) be the embedded fullshift in question. Let $\Gamma_0, \Gamma_1 \in \Xi$ be given with

$$\Gamma_0 \cap S = \emptyset \text{ and } \Gamma_1 \cap S = S.$$

Then we have

$$\liminf_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sharp(\Gamma_0 \cap A_n) \leq \liminf \frac{1}{\mathrm{Leb}(A_n)} \sharp((U \setminus S) \cap A_n) \leq \nu_U - \nu_S < \frac{\nu_U}{2}$$

but at the same time

$$\limsup_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sharp(\Gamma_1 \cap A_n) \geq \limsup_{n\to\infty} \frac{1}{\mathrm{Leb}(A_n)} \sharp(S \cap A_n) = \nu_S > \frac{\nu_U}{2}.$$

This contradicts the existence of uniform patch frequencies discussed above and thus excludes unique ergodicity.

To show the statement for the case of a fullshift contained in a fibre, note that for an embedded fullshift in a fibre the grid U is contained in $\curlywedge(W + h) - s$ for some $h \in H$ and $s \in \mathbb{R}^N$ (compare Lemma 5.2). By Theorem 4.13(ii) we then have

$$\nu_U \leq \frac{1}{\mu(\mathcal{L})} \Theta_H(W + h) = \frac{\Theta_H(W)}{\mu(\mathcal{L})}.$$

Now the statement follows from the considerations in the first part of the proof. ☐

5.4 Random Windows and Positive Entropy

In this section we will provide a probabilistic construction for a window which then yields almost surely a Delone dynamical system with positive topological entropy. As seen in the discussions of Section 5.1, the existence of an embedded fullshift is a sufficient condition for positive entropy. Further, the existence of such a fullshift follows from a certain structure of the window (for instance, from local topological independence). Using this dependence is a main ingredient for the following construction as well as the constructions in the following Sections 5.5 and 5.6.

In contrast to the discussions before we will focus on Euclidean CPS with Euclidean internal group $H = \mathbb{R}$. Recalling the discussions in Section 4.2, we then can represent $\mathcal{L}$ as $A(\mathbb{Z}^{N+1})$, where A is an irregular matrix. In particular, $\mu(\mathcal{L}) = \det(A)$ (compare Section 2.1).

For the construction, we assume $C \subsetneq [0, 1] \subseteq \mathbb{R}$ to be a Cantor set such that $\mathrm{Leb}(C) > 0$. Let $(G_n)_{n\in\mathbb{N}}$ be a numbering of the bounded connected components in $\mathbb{R}\setminus C$. We then define for $\omega \in \Sigma^+ = \{0, 1\}^{\mathbb{N}}$ the set

$$W(\omega) = C \cup \bigcup_{n\in\mathbb{N}:\omega_n=1} G_n.$$

Let $\mathbb{P}$ be the *Bernoulli measure* on Σ^+ with probability $p \in (0,1)$, i.e. $\mathbb{P}$ is the product measure on

$$\prod_{n\in\mathbb{N}} \mu,$$

where μ is the measure on $\{0,1\}$ which assigns the value p to $\{0\}$ and $1-p$ to $\{1\}$.

Lemma 5.15. *For $\mathbb{P}$-almost every $\omega \in \Sigma^+$, the window $W(\omega)$ is proper.*

Proof. First note that the complement of $W(\omega)$ in $\mathbb{R}$ consists of a union of connected components of $\mathbb{R}\setminus C$ (exactly those G_N with $\omega_n = 0$ and the two unbounded components in the complement of C). Since these are all open we obtain compactness of $W(\omega)$. By definition of $W(\omega)$ we obtain

$$\partial W(\omega) \subseteq \partial C \cup \bigcup_{n\in\mathbb{N}:\omega_n=1} \partial G_n = C.$$

Next, we are going to show that the reverse inclusion holds $\mathbb{P}$-almost surely. Suppose $x \in C$. Since C is perfect, there exists a sequence of gaps $(G_{n_k})_{k\in\mathbb{N}}$ such that

$$\inf G_{n_k} = \inf_{g\in G_{n_k}} g \to x \text{ as } k\to\infty.$$

By definition of the window, only intervals G_{n_k} with $\omega_{n_k} = 1$ are included in $W(\omega)$. Since all random variables are independent, the Borel-Cantelli-Lemma (compare for instance [Dur10]) implies

$$\mathbb{P}\left\{\text{for infinitely many } k \text{ the set } G_{n_k} \text{ is included in } W(\omega)\right\} = 1$$

as well as

$$\mathbb{P}\left\{\text{for infinitely many } k \text{ the set } G_{n_k} \text{ is not included in } W(\omega)\right\} = 1.$$

Thus, for $\mathbb{P}$-almost every ω there exist subsequences $G_{n_{k_j}} \subseteq W(\omega)$ and $G_{n_{k'_j}} \subseteq \mathbb{R}\setminus W(\omega)$ such that

$$\lim_{j\to\infty} \inf G_{n_{k_j}} = \lim_{j\to\infty} \inf G_{n_{k'_j}} = x.$$

Hence, we have $x \in \partial W(\omega)$ $\mathbb{P}$-almost surely for every fixed $x \in C$. Now let $M \subseteq C$ be a countable and dense subset of C. Then for any $x \in M$ the argument above shows $x \in \partial W(\omega)$ $\mathbb{P}$-almost surely. Hence, the countable set M is contained in $\partial W(\omega)$ $\mathbb{P}$-almost surely. Consequently, we also have

$$\mathrm{cl}(M) = C \subseteq \partial W(\omega)$$

$\mathbb{P}$-almost surely. Together with the other inclusion shown above, this yields

$$\partial W(\omega) = C$$

$\mathbb{P}$-almost surely. Form this we obtain

$$\mathrm{int}(W(\omega)) = W(\omega)\setminus \partial W(\omega) = \bigcup_{n\in\mathbb{N}:\omega_n=1} G_n$$

$\mathbb{P}$-almost surely. Using this equality and going again through the argument giving $C \subseteq \partial W(\omega)$, we then find $\mathbb{P}$-almost surely

$$C \subseteq \partial(\mathrm{int}(W(\omega))).$$

From this we then obtain

$$\mathrm{cl}(\mathrm{int}(W(\omega))) = \mathrm{int}(W(\omega)) \cup \partial(\mathrm{int}(W(\omega))) \supseteq \mathrm{int}(W(\omega)) \cup C = W(\omega)$$

and hence

$$\mathrm{cl}(\mathrm{int}(W(\omega))) = W(\omega)$$

for $\mathbb{P}$-almost every $\omega \in X$. $\square$

In the next step we need to find a suitable $\theta \in \mathbb{R}$ and a respective subset $S \subseteq \curlywedge(C + \theta)$ of positive asymptotic density. In order to avoid some technicalities later, it turns out convenient to work with

$$\tilde{C} = C \setminus \left(\bigcup_{n \in \mathbb{N}} \partial G_n \cup \{\inf C, \sup C\} \right).$$

Note that, since $C \setminus \tilde{C}$ is just the countable set of endpoints of intervals G_n together with the two extremal points of C, we have $\mathrm{Leb}(C) = \mathrm{Leb}(\tilde{C})$.

As a direct consequence of Theorem 4.13 we obtain the following.

Lemma 5.16. *For* Leb*-almost every* $\theta \in \mathbb{R}$ *the sequence* $\curlywedge(\tilde{C} + \theta)$ *has an asymptotic density given by*

$$\nu_{\curlywedge(\tilde{C}+\theta)} = \frac{\mathrm{Leb}(C)}{\det(A)}.$$

It remains to show that the random window $W(\omega)$ is $\mathbb{P}$-almost surely locally topologically independent in the sequence $\curlywedge(\tilde{C} + \theta)^*$.

Lemma 5.17. *Let C be a Cantor set with positive measure and let $W(\omega)$ be defined as above. Choose $\theta \in \mathbb{R}$. Then for $\mathbb{P}$-almost every $\omega \in \Sigma^+$ the window $W(\omega) + \theta$ is locally topologically independent in $L^* \cap (\tilde{C} + \theta)$.*

Proof. Let F be an arbitrary finite subset of $L^* \cap (\tilde{C} + \theta)$. Let

$$\delta_1 = \frac{1}{2} \min_{x \neq y \in F} |x - y|.$$

Since any Cantor set is nowhere dense and perfect, for $x \in F$ there exist gaps $I_1^x \subseteq (0, \delta_1)$ of $C + \theta - x$ such that

$$\bigcap_{x \in F} I_1^x \neq \emptyset.$$

By the choice of δ_1, we have $I_1^x \neq I_1^y$ if $x \neq y \in F$. Further, if we let

$$\delta_2 = \min \left\{ 1, \min_{x \in F} \inf I_1^x \right\},$$

then by the same argument there exist pairwise different gaps $I_2^x \subseteq (0, \delta_2)$ of $C + \theta - x$ such that

$$\bigcap_{x \in F} I_2^x \neq \emptyset.$$

Proceeding inductively with this construction, in the $(n+1)$-st step we define

$$\delta_{n+1} = \min \left\{ \frac{1}{n}, \min_{x \in F} \inf I_n^x \right\}$$

and choose gaps $I_{n+1}^x \subseteq (0, \delta_{n+1})$ of $C + \theta - x$ such that

$$\bigcap_{x \in F} I_{n+1}^x \neq \emptyset.$$

Now let $(G_n)_{n \in \mathbb{N}}$ be a labeling of all gaps of $C + \theta$. Then by construction we have

$$I_j^x = G_{n_j^x} - x \text{ for some } n_j^x \in \mathbb{N}.$$

Moreover, the choice of the δ_n and $I_n^x \subseteq (0, \delta_n)$ ensures that $n_j^x \neq n_{j'}^{x'}$ if $(x, j) \neq (x', j')$. In particular, this means that $\left(\omega_{n_j^x}\right)_{j \in \mathbb{N}, x \in F}$ is a two-parameter family of identically distributed independent random variables. Therefore, we obtain that for any $a \in \{0,1\}^F$ the set

$$\Omega(a) = \{\omega \in \Sigma^+ \mid \exists \text{ infinitely many } j \in \mathbb{N} : \omega_{n_j^x} = 1 \Longleftrightarrow a_x = 1\}$$

has full measure $\mathbb{P}(\Omega(a)) = 1$. However, for all $\omega \in \Omega(a)$ we have that

$$I_j = \bigcap_{x \in F} I_j^x \subseteq \left(\bigcap_{x \in F: a_x = 1} W(\omega) + \theta - x \right) \setminus \left(\bigcup_{x \in F: a_x = 0} W(\omega) + \theta - x \right).$$

Since the intervals I_j are all open and $\lim_{j \to \infty} \inf I_j = 0$, this shows the local topological independence of $W(\omega)$ in F. As this works for any finite subfamily F of $L^* \cap (\tilde{C} + \theta)$ and there exist only countably many such subfamilies, we obtain local topological independence of $W(\omega) + \theta$ in $L^* \cap (\tilde{C} + \theta)$ for $\mathbb{P}$-almost every $\omega \in \Sigma^+$. □

Now we may provide our main theorem regarding random windows which reads as follows.

Theorem 5.18. *Let $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ be a CPS with window $W(\omega)$ constructed as above. Then there exists a subset $\Sigma_0^+ \subseteq \Sigma^+$ of full $\mathbb{P}$-measure such that the following holds.*

(i) For all $\omega \in \Sigma_0^+$ and $h \in \mathbb{R}$ there exists a set $\Xi(\omega) \subseteq \mathbb{T}$ of full measure such that the Delone dynamical system $(\Omega(\curlywedge(W(\omega) + h)), \mathbb{R}^N)$ contains an embedded fullshift in $\beta_h^{-1}(\xi)$ for every $\xi \in \Xi(\omega)$.

(ii) For all $\omega \in \Sigma_0^+$ and $h \in \mathbb{R}$ the Delone dynamical system $(\Omega(\curlywedge(W(\omega) + h)), \mathbb{R}^N)$ has positive topological entropy

$$h_{\mathrm{top}}(\mathbb{R}^N) = \frac{1}{\det(A)} \mathrm{Leb}(C) \log 2.$$

(iii) For every $\omega \in \Sigma_0^+$ there exists a residual set $\mathcal{H} \subseteq \mathbb{R}$ such that the Delone dynamical system $(\Omega(\curlywedge(W(\omega) + h)), \mathbb{R}^N)$ is minimal for every $h \in \mathcal{H}$.

(iv) For every $\omega \in \Sigma_0^+$ and every $h \in \mathbb{R}$ the Delone dynamical system $(\Omega(\curlywedge(W(\omega) + h)), \mathbb{R}^N)$ is not uniquely ergodic provided that C additionally satisfies $\mathrm{Leb}(C) > \frac{1}{2}$.

Proof. By Lemma 5.15 there exists as set Σ_1^+ of full measure in Σ^+ such that $W(\omega)$ is proper for every $\omega \in \Sigma_1^+$. By Lemma 5.17 there exists a set Σ_2^+ of full measure in Σ^+ such that for every $\omega \in \Sigma_2^+$ the window $W(\omega) + \theta$ is locally topologically independent in $L^* \cap (\tilde{C} + \theta)$ some $\theta \in \mathbb{R}$. By Fubini's Theorem this statement for local topological independence holds for Leb-almost every $\theta \in \mathbb{R}$. Then

$$\Sigma_0^+ = \Sigma_1^+ \cap \Sigma_2^+$$

is a set of full measure in Σ^+. Now consider an arbitrary $\omega \in \Sigma_0^+$.

(i). As due to Lemma 5.16 the set $\curlywedge(\tilde{C} + \theta)$ has asymptotic density $\frac{\mathrm{Leb}(C)}{\det(A)}$ for almost every $\theta \in \mathbb{R}$, we obtain that for almost every $\theta \in \mathbb{R}$ the assumptions of Lemma 5.12 are satisfied for $W = W(\omega) + \theta$ and $S = \curlywedge(\tilde{C} + \theta)$. Applying Lemma 5.12 we then obtain a set $\Xi'(\omega) \subseteq \mathbb{R}$ of full measure such that for any $\theta \in \Xi'(\omega)$ and any $h \in \mathbb{R}$ there exists an embedded fullshift in the fibre $\beta_h^{-1}(\xi)$ for $\xi = [0, \theta - h]_{\mathcal{L}} \in \mathbb{T}$. Since the existence of an embedded fullshift in a fibre is a property that is invariant under translation (compare Lemma 5.1), we then obtain an embedded fullshift for all $[t, \theta - h]_{\mathcal{L}}$ with $(t, \theta) \in \mathbb{R}^N \times \Xi'(\omega)$. The projection of the latter set to $\mathbb{T}$ gives the required full measure set $\Xi(\omega)$ which satisfies the assertion (i).

(ii). We note that the proven part (i) together with Lemma 5.7 directly yields

$$h_{\mathrm{top}}(\mathbb{R}^N) \geq \frac{1}{\det(A)} \mathrm{Leb}(C) \log 2.$$

On the other hand, by Lemma 4.41 we know that

$$h_{\mathrm{top}}(\mathbb{R}^N) \leq \frac{1}{\det(A)} \mathrm{Leb}(C) \log 2$$

which implies the assertion.

(iii). This statement is a direct consequence of Lemma 4.11, Lemma 4.12(i) and Proposition 4.25(i).

(iv). The preceding considerations give almost surely an embedded fullshift with set of free points S satisfying

$$\nu_S = \frac{1}{\det(A)}\mathrm{Leb}(C).$$

Since $C \subseteq [0,1]$, the grid U must be contained in $⅄([0,1]+h)-t$ for some $h, t \in \mathbb{R}$. Thus,

$$\nu_U \leq \nu_{⅄([0,1]+h)-t} \leq \frac{1}{\det(A)},$$

where the last inequality holds due to Theorem 4.13. This yields

$$\nu_S > \frac{\nu_U}{2}$$

and Proposition 5.14 gives the desired statement. ☐

As an immediate consequence of Lemma 4.37 we obtain the following.

Corollary 5.19. *Suppose the situation of Theorem 5.18. Whenever $W(\omega)$ is proper, the dynamical system $(\Omega(⅄(W(\omega)+h)), \mathbb{R}^N)$ has the torus $\mathbb{T}$ as its maximal equicontinuous factor.*

5.5 A deterministic construction for CPS with Positive Entropy

In contrast to the construction in the section before we now will provide a deterministic construction of proper model sets with positive topological entropy. In the following section we are going to extend this construction to the case of weak model sets. The starting point of our construction will be the construction of an initial Cantor set C_0 that is adapted to the respective CPS. To that end, we need to introduce some further notation.

Consider the Euclidean CPS $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$. As already pointed out in Section 4.2, there exists an irrational matrix $A = (a_{ij})_{i,j=1}^{N+1} \in \mathrm{GL}(N+1, \mathbb{R})$ such that $\mathcal{L} = A(\mathbb{Z}^{N+1})$. Without loss of generality we may assume that $a_{N+1,j} \in (0,1)$ for all $j = 1, \ldots, N$ and $a_{N+1,N+1} = 1$. In this case, given any $v = (v_1, \ldots, v_N) \in \mathbb{Z}^N$, there exists a unique $v_{N+1} \in \mathbb{Z}$ such that

$$l_v^* = \pi_{\mathbb{R}}\left(A\begin{pmatrix} v \\ v_{N+1}\end{pmatrix}\right) = \sum_{j=1}^{N+1} a_{N+1,j}v_j \in [0,1) \cap L^*.$$

By $a_{N+1,N+1} = 1$ we obtain in particular

$$l_v^* = \sum_{j=1}^{N} a_{N+1,j}v_j \mod 1.$$

Given $v \in \mathbb{Z}^N$, let $||v|| = \max_{j=1}^{N} |v_j|$ and fix a numbering $(v(n))_{n\in\mathbb{N}}$ of $\mathbb{Z}^N$ such that $||v(n)||$ is non-decreasing in n.

For $t \in \mathbb{N}$ let $\mathcal{N}_t = \mathbb{Z}^N \cap A_t$ and $R_t = \left[1, (2t+1)^N\right] \cap \mathbb{Z}$. Then the definition of the numbering implies

$$v(R_t) = \{v(n) \mid n \in R_t\} = \mathcal{N}_t.$$

Lemma 5.20. *There exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ such that the open intervals $I_k = \left(l^*_{v(n_k)}, l^*_{v(n_k)} + \varepsilon_k\right)$ satisfy*

(i) $I_j \cap I_k = \emptyset$ *for all* $j \neq k$.

(ii) $\mathrm{cl}\left(\bigcup_{k\in\mathbb{N}} I_k\right) = [0,1]$.

(iii) $\lim_{k\to\infty} \frac{k}{n_k} > \frac{1}{2}$.

Proof. For simplicity, we work on the additive group $\mathbb{R}/\mathbb{Z}$ and omit to write $\mod 1$. Hence, by slightly abusing notation we automatically interpret real numbers as elements of the circle. In particular, we denote by $d(x,0)$ the distance of $x \in \mathbb{R}$ to the nearest integer. We choose a strictly increasing sequence of integers $(\kappa(t))_{t\in\mathbb{N}}$ that satisfies

$$\sum_{t\in\mathbb{N}} \frac{\sharp\mathcal{N}_{2t}}{\sharp\mathcal{N}_{\kappa(t)}} \leq \frac{1}{2\cdot 5^N}. \tag{5.5.1}$$

For $v \in \mathbb{Z}^N$ set

$$\eta_v = \frac{1}{2}\min\left\{d\left(l_u^*, 0\right) \mid u \in \mathcal{N}_{\kappa(||v||)} \cap \mathbb{Z}^N\right\}.$$

Now let

$$J_n = \left[l^*_{v(n)}, l^*_{v(n)} + \eta_{v(n)}\right) \cap [0,1].$$

We then define the set

$$B = \{n \in \mathbb{N} \mid J_n \cap J_j \neq \emptyset \text{ for } j < n\}.$$

We now want to estimate the cardinality of $B \cap R_t$. To that end, note that if $J_n \cap J_j \neq \emptyset$ and $J_{n'} \cap J_j \neq \emptyset$ for some $n, n' > j$, then

$$d\left(l^*_{v(n)}, l^*_{v(n')}\right) = d\left(l^*_{v(n)-v(n')}, 0\right) < 2\eta_{v(j)}$$

and therefore $v(n) - v(n') \notin \mathcal{N}_{\kappa(||v(j)||)}$. Similarly, $v(n) - v(j), v(n') - v(j) \notin \mathcal{N}_{\kappa(||v(j)||)}$. Covering $\mathcal{N}_t \setminus \left(\mathcal{N}_{\kappa(||v(j)||)} + v(j)\right)$ by at most

$$\frac{\sharp\mathcal{N}_{2t}}{\sharp\mathcal{N}_{\kappa(||v(j)||)}}$$

translates of $\mathcal{N}_{\kappa(||v(j)||)}$ for each j leads to the following rough estimate.

$$\begin{aligned}
\sharp(B \cap R_t) &\leq \sum_{j=1}^{\sharp\mathcal{N}_t} \sharp\left\{n \in \{j+1, \ldots, \sharp\mathcal{N}_t\} \mid J_n \cap J_j \neq \emptyset\right\} \\
&\leq \sum_{j=1}^{\sharp\mathcal{N}_t} \frac{\sharp\mathcal{N}_{2t}}{\sharp\mathcal{N}_{\kappa(||v(j)||)}} \leq \sharp\mathcal{N}_{2t} \sum_{k=1}^{t} \frac{\sharp\mathcal{N}_k}{\sharp\mathcal{N}_{\kappa(k)}} \overset{(5.5.1)}{\leq} \frac{\sharp\mathcal{N}_{2t}}{2\cdot 5^N} \leq \frac{\sharp\mathcal{N}_t}{2}.
\end{aligned}$$

Now let $n_1 = 0$ and define

$$n_{k+1} = \min\{n > n_k \mid J_n \cap J_{n_j} = \emptyset \text{ for all } j \leq k\}.$$

Then by defining $\varepsilon_k = \min\left\{\eta_{v(n_k)}, 1 - l^*_{v(n_k)}\right\}$ and thus $I_k = J_{n_k}$, property (i) follows by construction.

Assume, the union over all I_k would not be dense in $[0,1]$. Then there exists some interval $(\alpha, \beta) \subseteq [0,1]$ which does not intersect any of the I_k. Then any interval J_n that is contained in (α,β) appears as some I_k by the above construction, which leads to a contradiction. Note that, by definition of Cut and Project Schemes, the image of $v \mapsto l_v^*$ is dense in $[0,1]$, so there exists some $n \in \mathbb{N}$ such that $J_n \subseteq (\alpha,\beta)$. Thus, (ii) holds.

Now let $\mathcal{N} = \{n_k \mid k \in \mathbb{N}\}$. Then we have

$$B' = \mathbb{N} \setminus \mathcal{N} \subseteq B.$$

Thus, we have

$$\sharp(\mathcal{N} \cap R_t) \geq \sharp R_t - \sharp(R_t \cap B) \geq \frac{\sharp R_t}{2}.$$

By using

$$\lim_{t\to\infty} \frac{\sharp(R_t \setminus R_{t-1})}{\sharp R_t} = \lim_{t\to\infty} \frac{\sharp(\mathcal{N}_t \setminus \mathcal{N}_{t-1})}{\sharp\mathcal{N}_t} = 0,$$

this yields that

$$\lim_{m\to\infty} \frac{\sharp(\mathcal{N} \cap \{1,\dots,m\})}{m} \geq \frac{1}{2}.$$

Hence (iii) is proven. $\square$

Since all intervals I_k are pairwise disjoint and their union is dense in $[0,1]$, the set

$$C_0 = [0,1] \setminus \bigcup_{k\in\mathbb{N}} I_k$$

is a Cantor set. Further, property (iii) of the lemma above implies that the Lebesgue-measure of C_0 is positive.

Lemma 5.21. *Let C be a Cantor set in $[0,1]$ such that $\{0,1\} \subseteq C$. Then there exists a sequence of open sets $O_j \subseteq [0,1]$ such that*

(i) For all $j \in \mathbb{N}$ the set O_j is a union of gaps of C.

(ii) $\partial O_j = C$ for all $j \in \mathbb{N}$.

(iii) The family $(O_j)_{j\in\mathbb{N}}$ is locally topologically independent in 0.

Proof. It is well-known that for any two Cantor sets $C, C' \subseteq [0,1]$ with $\{0,1\} \subseteq C \cap C'$ there exists an orientation-preserving homeomorphism of $[0,1]$ which maps C to C'. So without loss of generality we may assume C is the middle third Cantor set. Then we can write

$$C = \left\{ \sum_{n=1}^{\infty} 2a_n 3^{-n} : a \in \{0,1\}^{\mathbb{N}} \right\}.$$

Denote by $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ the set of finite words with letters 0 and 1 and denote by $|a|$ the length of a word $a \in \mathcal{A}$. Then

$$G_a = \left(\sum_{n=1}^{|a|} 2a_n 3^{-n} + 3^{-n}, \sum_{n=1}^{|a|} 2a_n 3^{-n} + 2 \cdot 3^{-|a|} \right) \tag{5.5.2}$$

are exactly the gaps of C. We will construct the sets O_j such that they all contain

$$O = \bigcup_{a\in\mathcal{A}:|a|\in 4\mathbb{N}} G_a$$

but no G_a with $|a| \in 4\mathbb{N}+1$. Since all points of C are approximated by gaps of both types we always have $\partial O_j = C$. Thus, properties (i) and (ii) hold.

Let $a^{(n)} = 0^{2n+1}1 \in \{0,1\}^{2n+2}$. Choose a countable partition $(S_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ into infinite sets. Further, let $(M_j, N_j)_{j\in\mathbb{N}}$ be a numbering of all pairs of disjoint finite sets of integers. Then let

$$V_j = \bigcup_{n\in\mathbb{N}: j\in M_n} S_n$$

and

$$O_j = O \cup \bigcup_{l\in V_j} G_{a^{(l)}}.$$

For any $n \in \mathbb{N}$ the set S_n is a subset of V_j whenever $j \in M_n$, and S_n is disjoint from all V_j whenever $j \in N_n$. Thus, the set

$$\bigcap_{j\in M_n} O_j \setminus \bigcup_{j\in N_n} O_j$$

contains $\bigcup_{l\in S_n} G_{a^{(l)}}$. Since S_n is infinite, this shows the local topological independence required in (iii). $\square$

Now let $C_0 = [0,1] \setminus \bigcup_{k\in\mathbb{N}} I_k$ as above and define a window $W \subseteq \mathbb{R}$ by

$$W = C_0 \cup \bigcup_{k\in\mathbb{N}} (I_k \cap \mathrm{cl}(O_k + \inf(I_k))).$$

Note that by construction we have $\inf(I_k) = l^*_{n_k}$.

Lemma 5.22. *The window W constructed above is proper.*

Proof. By Lemma 5.21 we have $\partial W = C_0$. Hence,

$$\mathrm{int}(W) = \bigcup_{k\in\mathbb{N}} I_k \cap \mathrm{cl}(O_k + \inf(I_k)).$$

On the other hand, a straightforward calculation yields

$$\bigcup_{k\in\mathbb{N}} I_k \cap \mathrm{cl}(O_k + \inf(I_k)) = \bigcup_{k\in\mathbb{N}} I_k \cap W.$$

By definition of W we have $\mathrm{cl}\left(\bigcup_{k\in\mathbb{N}} I_k \cap W\right) = W$. Thus, the assertion holds. □

Theorem 5.23. *Let $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ be a CPS with window W constructed as above. Choose $S = \left(l_{v(n_k)}\right)_{k\in\mathbb{N}}$, where $(n_k)_{k\in\mathbb{N}}$ is chosen as in Lemma 5.20 and $l_{v(n_k)}$ is defined by $\left(l_{v(n_k)}, l^*_{v(n_k)}\right) \in \mathcal{L}$. Then the following holds.*

(i) For all $h \in \mathbb{R}$ the pair (Ξ, S) with

$$\Xi = \beta_h^{-1}([0,-h]_{\mathcal{L}})$$

is an embedded fullshift with grid $U = \curlywedge(W)$. In particular, the system $(\Omega(\curlywedge(W + h)), \mathbb{R}^N)$ has positive topological entropy for all $h \in \mathbb{R}$.

(ii) For all $h \in \mathbb{R}$ the system $(\Omega(\curlywedge(W + h)), \mathbb{R}^N)$ is not uniquely ergodic.

Proof. (i). By construction, the local topological independence of W in S^* is equivalent to the local topological independence of the sets $(O_k)_{k\in\mathbb{N}}$ and thus follows from Lemma 5.21(iii). Hence, by Lemma 5.12 $\beta_h^{-1}([0,-h]_{\mathcal{L}})$ contains an embedded fullshift.

(ii). Let $U' = \{l_v \mid v \in \mathbb{Z}^N\} = \curlywedge([0,1])$. By applying Theorem 4.13 we then obtain

$$\nu_{U'} = \frac{1}{\det(A)}.$$

As $U \subseteq U'$ we have that

$$\nu_U \le \frac{1}{\det(A)}.$$

At the same time it follows from Lemma 5.20(iii) that

$$\nu_S \ge \frac{\nu_{U'}}{2} = \frac{1}{2\det(A)} \ge \frac{\nu_U}{2}.$$

Hence, (ii) follows by Proposition 5.14. □

5.6 Weak Model Sets with Positive Entropy

As mentioned before, in this section we will modify the construction of the previous section such that the resulting window has empty interior, but the corresponding dynamical hull still provides positive topological entropy. Note that in this case we are not dealing with Delone sets anymore.

Lemma 5.24. *Let $C \subseteq [0,1]$ be the middle third Cantor set. Then there exists a sequence of sets $O_j \subseteq [0,1]$ such that* *(i)* $C \subseteq \partial O_j$ *for all* $j \in \mathbb{N}$.

(ii) $\operatorname{int}(O_j) = \emptyset$ *for all* $j \in \mathbb{N}$.

(iii) The family $(O_j)_{j\in\mathbb{N}}$ *is locally metrically independent in* 0.

Proof. We can write

$$C = \left\{ \sum_{n=1}^{\infty} 2a_n 3^{-n} : a \in \{0,1\}^{\mathbb{N}} \right\}.$$

As before, let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$, denote by $|a|$ the length of $a \in \mathcal{A}$ and by G_a the gap of C corresponding to a. Let $K \subseteq [0,1]$ be another Cantor set such that $\{0,1\} \subseteq K$, $\operatorname{Leb}(K) > 0$ and $0 \in \{x \in \mathbb{R} \mid \operatorname{Leb}(B_\varepsilon(0) \cap K) > 0 \text{ for all } \varepsilon > 0\}$. We will construct the sets O_j such that each set contains C and, to ensure metric independence, we insert K into the gaps of C. Thus, let again $a^{(n)} = 0^{2n+1}1 \in \{0,1\}^{2n+2}$ and choose a countable partition $(S_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ into infinite sets. Further, let $(M_j, N_j)_{j\in\mathbb{N}}$ be a numbering of all pairs of disjoint finite sets of integers. Then let

$$V_j = \bigcup_{n\in\mathbb{N}: j\in M_n} S_n$$

and

$$O_j = C \cup \bigcup_{l\in V_j} G_{a^{(l)}} \cap (K + \inf G_{a^{(l)}}).$$

Then conditions (i) and (ii) follow directly by construction. Further, for any $n \in \mathbb{N}$ the set S_n is a subset of V_j whenever $j \in M_n$ and S_n is disjoint from V_j whenever $j \in N_n$.

Since S_n is infinite, for any $\varepsilon > 0$ there exists $l \in S_n$ such that $G_{a^{(l)}} \subseteq B_\varepsilon(0)$. Since $0 \in \{x \in \mathbb{R} \mid \operatorname{Leb}(B_\varepsilon(0) \cap K) > 0 \text{ for all } \varepsilon > 0\}$, the set $G_{a^{(l)}} \cap (K + \inf G_{a^{(l)}})$ has positive measure. Thus, as

$$G_{a^{(l)}} \cap (K + \inf G_{a^{(l)}}) \subseteq B_\varepsilon(0) \cap \left(\bigcap_{j\in M_n} O_j \setminus \bigcup_{j\in N_n} O_j \right)$$

(and hence $G_{a^{(l)}} \cap (K + \inf G_{a(l)}) \subseteq \bigcap_{j\in M_n} O_j \setminus \bigcup_{j\in N_n} O_j$), the set on the right side has positive measure. Since this holds for all $\varepsilon > 0$ and (M_n, N_n) was arbitrary, this shows the metric independence of the family $(O_j)_{j\in\mathbb{N}}$. ☐

Now let $(n_k)_{k\in\mathbb{N}}$ and the intervals I_k be as in Lemma 5.20. As in the previous section define

$$C_0 = [0,1] \setminus \bigcup_{k\in\mathbb{N}} I_k.$$

Then

$$W = C_0 \cup \bigcup_{k\in\mathbb{N}} (I_k \cap (O_k + \inf I_k))$$

is a compact subset of $[0,1]$ such that $\operatorname{int}(W) = \emptyset$. Again, we have $\inf I_k = l^*_{v(n_k)}$ by construction of the I_k. Similar to Theorem 5.23 we obtain the following.

Theorem 5.25. *Let* $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ *be a CPS with compact window* W *as defined above. Further, choose* $S = (l_{n_k})_{k\in\mathbb{N}}$ *as in Theorem 5.23. Then for* Leb*-almost all* $h \in \mathbb{R}$ *the hull* $\Omega(\curlywedge(W+h))$ *contains an embedded fullshift and the dynamical system* $(\Omega(\curlywedge(W+h)), \mathbb{R}^N)$ *has positive topological entropy.*

Proof. By construction, the metric independence of W in S^* is equivalent to the metric independence of the sets $(O_k)_{k\in\mathbb{N}}$ and thus follows from Lemma 5.24(iii). Hence, by Lemma 5.13, $\Omega(\curlywedge(W+h))$ contains an embedded fullshift for Leb-almost all $h \in \mathbb{R}$. ☐

Remark 5.26. (i) Since $\operatorname{int}(W) = \emptyset$ we have $\emptyset \in \Omega(\curlywedge(W+h))$. Thus, it is obvious that the system cannot be uniquely ergodic.

(ii) Similar as in Lemma 5.12 one may show that the obtained embedded fullshift Ξ is contained in $人(W + h)$. In the case of proper model sets, this was used further to conclude that Ξ is contained in a certain fibre. However, for weak model sets there is no analogous statement to that, since the torus parametrization does not exist in this case.

Chapter 6

Irregular Cut and Project Schemes with Zero Entropy

In the previous chapter we discussed irregular Euclidean Cut and Project Schemes where the corresponding hull has positive topological entropy. However, the implication

$$\text{"irregularity"} \Rightarrow \text{"positive topological entropy"}$$

does not hold in general. As mentioned in the Introduction, the authors of [BJL16] constructed an irregular CPS with integers as physical space and an odometer as internal space such that the above implication does not hold. In this chapter we are going to construct counterexamples in Euclidean space. Additionally, we provide a class of irregular windows which yield uniquely ergodic dynamical hulls.

6.1 Criteria for Zero Entropy

Let G and H be locally compact abelian second-countable groups. Consider a CPS $(G, H, \mathcal{L})$ with proper window $W \subseteq H$ and torus parametrization $\beta : \Omega(\curlywedge(W)) \to \mathbb{T}$. In this section, we provide two criteria in terms of the local structure of W which control the structure of the fibres of β and their entropy. We would like to point out the following things:

- Although our examples later will be constructed in Euclidean space, we provide the mentioned criteria in a more general setting.
- We note that all $[g, h]_{\mathcal{L}} \in \mathbb{T}$ are translates of $[0, h]_{\mathcal{L}}$. Hence, $\sharp\beta^{-1}([g, h]_{\mathcal{L}}) = \sharp\beta^{-1}([0, h]_{\mathcal{L}})$. Therefore, throughout this chapter, it will be sufficient to consider points $[0, h]_{\mathcal{L}} \in \mathbb{T}$.

The first criterion for vanishing entropy will make use of the cardinality of the fibres of β. We will require the window to satisfy a certain condition on "self similarity". This will ensure that all fibres of β have finite cardinality, which directly implies that the fibres won't carry any entropy.

Recalling the discussions in Section 3.6 and Theorem 3.18 we immediately obtain

Lemma 6.1. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. Then the following holds.*

(i) If $\sharp\beta^{-1}(\xi) < \infty$ for all $\xi \in \mathbb{T}$, then $h^{\xi}_{\mathrm{top}}(\Omega(\curlywedge(W)), G) = 0$.

(ii) If $G = \mathbb{R}$ and $\sharp\beta^{-1}(\xi) < \infty$ for all $\xi \in \mathbb{T}$, then $h_{\mathrm{top}}(\Omega(\curlywedge(W)), G) = 0$.

Remark 6.2. Note that the assumptions in this lemma are quite strong, since we require all fibres to be finite. As seen in [MP79] or Corollary 4.36, for regular windows we only have $\sharp\beta^{-1}(\xi) = 1$ for $\Theta_{\mathbb{T}}$-almost all $\xi \in \mathbb{T}$. Thus, in case $\Theta_H(\partial W) = 0$ there may exist fibres with infinite cardinality.

Therefore, it is our first goal to control the number of elements in the fibres of β. Thus, let $[0,h]_{\mathcal{L}} \in \mathbb{T}$. As seen in Lemma 4.34, Delone sets contained in $\beta^{-1}([0,h]_{\mathcal{L}})$ basically differ from each other in points $l \in G$ whose conjugates l^* are contained in $\partial W + h$. Hence, we have to distinguish two cases.

(i) If $\partial W + h \cap L^* = \emptyset$, then $\sharp\beta^{-1}([0,h]_{\mathcal{L}}) = 1$. Hence, the fibre carries no entropy.

(ii) If $\partial W + h \cap L^* \neq \emptyset$, then the cardinality of $\{l^* \mid l^* \in \partial W + h \cap L^*\}$ may be finite or not.

Lemma 6.1 ensures that the respective fibre carries no entropy in case of

$$\sharp\{l^* \mid l^* \in \partial W + h \cap L^*\} < \infty.$$

Hence, we only have to investigate the infinite case.

We say a point $[0,h]_{\mathcal{L}} \in \mathbb{T}$ is *critical* if $\sharp\{l^* \mid l^* \in \partial W + h \cap L^*\} = \infty$. For a given critical point $[0,h]_{\mathcal{L}} \in \mathbb{T}$, we say that $l_1^*, l_2^* \in \partial W + h \cap L^*$ are *similar with respect to* h if there exists some $\varepsilon > 0$ such that

$$(B_\varepsilon(l_1^*) \cap (W+h)) - l_1^* = (B_\varepsilon(l_2^*) \cap (W+h)) - l_2^*.$$

It is easy to see that the following holds.

Corollary 6.3. *Let $h \in H$. Then*

$$l_1^* \sim_h l_2^* \iff l_1^*, l_2^* \textit{ are similar with respect to } h$$

defines an equivalence relation on $\partial W + h \cap L^*$.

If for every $h \in H$ there are only finitely many equivalence classes with respect to $\sim_h$, then we call W *self similar*.

The following lemma gives insight into the connection between self similar points with respect to some $h \in H$ and the structure of Delone sets in the respective fibre. Roughly spoken, points are contained in the same Delone set of $\beta^{-1}([0,h]_{\mathcal{L}})$ if their images under the star map are similar with respect to h.

Lemma 6.4. *Let $[0,h]_{\mathcal{L}} \in \mathbb{T}$. Suppose $l_1^* \sim_h l_2^*$. Then for each $\Gamma \in \beta^{-1}([0,h]_{\mathcal{L}})$ we have $l_1 \in \Gamma$ if and only if $l_2 \in \Gamma$.*

Proof. Fix $h \in H$. Let $\Gamma \in \beta^{-1}([0,h]_{\mathcal{L}})$ and suppose $l_1 \in \Gamma$. Due to Lemma 4.33 there exists a sequence $h_j \in L^*$ such that

- $\lim_{j\to\infty} h_j = h$,
- $l_1 \in \curlywedge(W + h_j)$ for all $j \in \mathbb{N}$.

This implies $l_1^* \in W + h_j$ for all $j \in \mathbb{N}$. Since $l_1^* \sim_h l_2^*$, there exists some $j_0 \in \mathbb{N}$ such that $l_2^* \in W + h_j$ for all $j > j_0$. Hence, $l_2 \in \lim_{j\to\infty} \curlywedge(W + h_j) = \Gamma$. The other implication follows by symmetry. ☐

An immediate consequence of this lemma is, that, for given $[0,h]_{\mathcal{L}} \in \mathbb{T}$, similar points with respect to h are (not) contained in the same Delone set. Thus, if there are only finitely many equivalence classes with respect to h, the corresponding fibre contains only finitely elements, which implies vanishing entropy.

Proposition 6.5. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. If W is self similar, we have $h^\xi_{\mathrm{top}}(\Omega(\curlywedge(W)), G) = 0$ for all $\xi \in \mathbb{T}$.*

Proof. Fix an arbitrary $\xi = [0, h]_{\mathcal{L}} \in \mathbb{T}$. Without loss of generality we may assume ξ to be critical, otherwise the claim holds by our considerations above. By self similarity of W we may decompose $\partial W + h \cap L^*$ into finitely many equivalence classes $E_i^*(h)$, i.e.

$$\partial W + h \cap L^* = \bigcup_{i=1}^{K(h)} E_i^*(h).$$

This observation and Lemma 6.4 immediately yield that each fibre has to be finite, i.e.

$$\sharp \beta^{-1}(\xi) \leq 2^{K(h)}.$$

Then, by Lemma 6.1, we obtain $h_{\text{top}}^{\xi}(\Omega(⅄(W)), G) = 0$. □

As a second criterion for vanishing entropy, we introduce a concept which still allows fibres to contain infinitely many elements. However, in this case these elements will have a certain geometrical structure which allows us to control the entropy in terms of the windows structure.

We say a window $W \subseteq H$ has *locally disjoint complements* if for all critical $[0, h]_{\mathcal{L}} \in \mathbb{T}$ and $l_1^*, l_2^* \in \partial W + h \cap L^*$ there exists $\varepsilon > 0$ such that

$$((B_\varepsilon(l_1^*) \cap (W + h)^c) - l_1^*) \cap ((B_\varepsilon(l_2^*) \cap (W + h)^c) - l_2^*) = \emptyset.$$

As we will show in the following lemma, for given $\xi \in \mathbb{T}$, all elements in $\beta^{-1}(\xi)$ will differ in at most one point from each other.

Lemma 6.6. *Suppose $(G, H, \mathcal{L})$ is a CPS with proper window $W \subseteq H$ such that W has locally disjoint complements. Then for all critical $\xi = [0, h]_{\mathcal{L}} \in \mathbb{T}$ there exists some $\Gamma_+ \in \beta^{-1}(\xi)$ such that for all $\Gamma \in \beta^{-1}(\xi)$ we have*

(i) $\Gamma \subseteq \Gamma_+$.

(ii) Γ_+ *differs from* Γ *in at most one point.*

Proof. Fix a critical $\xi = [0, h]_{\mathcal{L}} \in \mathbb{T}$ and let $l_0^* \in \partial W + h \cap L^*$. By Lemma 4.33 and Lemma 4.34 there exists $\Gamma' \in \beta^{-1}(\xi)$ such that $l_0 \notin \Gamma'$ and a sequence $h_j' \in L^*$ such that

- $\lim_{j \to \infty} h_j' = h$,
- $\Gamma' = \lim_{j \to \infty} ⅄(W + h_j')$,
- $l_0^* \in (W + h_j')^c$ for all $j \in \mathbb{N}$.

Now let $l^* \in (\partial W + h \cap L^*) \setminus \{l_0^*\}$. Since W has locally disjoint complements there exists some $\varepsilon > 0$ such that

$$0 \in \left(B_\varepsilon(l_0^*) \cap (W + h_j')^c\right) - l_0^* \Rightarrow 0 \notin \left(B_\varepsilon(l^*) \cap (W + h_j')^c\right) - l^*$$

for large enough j. Hence, for sufficiently large j we have $l^* \in W + h_j'$. This implies $l \in \lim_{j \to \infty} ⅄(W + h_j') = \Gamma'$.

As l^* was arbitrary, the above yields the existence of a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ in $\beta^{-1}(\xi)$ such that

$$\{l \mid l^* \in \partial W + h \cap L^*\} \cap B_n(0) \subseteq \Gamma_n.$$

Compactness of the fibres $\beta^{-1}(\xi)$ then gives a convergent subsequence with limit Γ_+ which verifies (i). Then (ii) follows immediately. □

Regarding the entropy, we obtain the following useful criterion.

Proposition 6.7. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. If W has locally disjoint complements, we have $h_{\text{top}}^{\xi}(\Omega(⅄(W)), G) = 0$ for all $\xi \in \mathbb{T}$.*

Proof. Let $\xi = [0, h]_{\mathcal{L}} \in \mathbb{T}$ be critical and $(A_n)_{n\in\mathbb{N}}$ a van Hove sequence in G. By Lemma 6.6 there exists $\Gamma_+ \in \beta^{-1}(\xi)$ such that every other set $\Gamma \in \beta^{-1}(\xi) \setminus \{\Gamma_+\}$ differs from Γ_+ in one point. We denote this point by $l(\Gamma)$. By $K_\cdot$ we denote the closed ball in G centered around the origin with radius $\cdot$. For $\varepsilon > 0$ and $n \in \mathbb{N}$ we define

$$\begin{aligned} S(G, \varepsilon, n) &= \left\{\Gamma \in \beta^{-1}(\xi) \mid l(\Gamma) \in K_{1/\varepsilon} + A_n\right\} \cup \{\Gamma_+\} \\ &= \left\{\Gamma \in \beta^{-1}(\xi) \mid l(\Gamma) \in \partial^{K_{1/\varepsilon}}(A_n) \cup A_n\right\} \cup \{\Gamma_+\} \end{aligned}$$

Then we obtain

$$\beta^{-1}(\xi) = \bigcup_{\Gamma \in S(G,\varepsilon,n)} \left\{\Gamma' \in \beta^{-1}(\xi) : \max_{s\in A_n} d(s\cdot\Gamma, s\cdot\Gamma') < \varepsilon\right\},$$

which implies that $S(G, \varepsilon, n)$ is an (ε, n)-spanning set.

Let $r = \min_{l\neq l'\in\Gamma_+} d_G(l, l')/2$. Then

$$\sharp S(G, \varepsilon, n) \leq \frac{1}{\Theta_G(K_r)} \Theta_G\left(\partial^{K_{1/\varepsilon+r}}(A_n) \cup A_n\right).$$

This yields

$$\begin{aligned} h_\varepsilon^\xi(\Omega(\curlywedge(W)), G) &\leq \limsup_{n\to\infty} \frac{1}{\Theta_G(A_n)} \log\left(\frac{1}{\Theta_G(K_r)}\Theta_G(\partial^{K_{1/\varepsilon+r}}(A_n) \cup A_n)\right) \\ &\leq \limsup_{n\to\infty} \frac{1}{\Theta_G(A_n)} \log\left(\frac{1}{\Theta_G(K_r)}\right) + \frac{1}{\Theta_G(A_n)}\left(\log\Theta_G(\partial^{K_{1/\varepsilon+r}}(A_n)) + \log\Theta_G(A_n)\right). \end{aligned}$$

Since A_n is a van Hove sequence we obtain that

$$\lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \log\Theta_G(\partial^{K_{1/\varepsilon+r}}(A_n)) = 0.$$

This yields $h_\varepsilon^\xi(\Omega(\curlywedge(W)), G) = 0$ and hence $h_{\mathrm{top}}^\xi(\Omega(\curlywedge(W)), G) = 0$. ☐

Due to the special geometrical structure of the elements of the fibres, it is even possible to get a statement regarding unique ergodicity.

Lemma 6.8. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$. If W has locally disjoint complements, the dynamical hull $(\Omega(\curlywedge(W)), G)$ is uniquely ergodic.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a tempered van Hove sequence in G (note that, by Lemma 2.12 and Lemma 2.13, G always admits such a sequence). Suppose there exist two invariant ergodic measures μ_1 and μ_2 on $\Omega(\curlywedge(W))$. Given $f \in C(\Omega(\curlywedge(W)))$ and $i \in \{1, 2\}$, the pointwise ergodic theorem 3.21 yields a subset $\Omega_i^f \subseteq \Omega(\curlywedge(W))$ of full μ_i-measure such that for all $\Gamma \in \Omega_i^f$ we have

$$\lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \int_{A_n} f(\Gamma - s)\, \mathrm{d}\Theta_G(s) = \int_{\Omega(\curlywedge(W))} f\, \mathrm{d}\mu_i. \tag{6.1.1}$$

We want to show that (6.1.1) holds for all $\Gamma \in \beta^{-1}(\beta(\Omega_i^f))$. To that end, given $g_0 \in G$ and $\varepsilon > 0$, consider the function

$$f_{g_0,\varepsilon} : \Omega(\curlywedge(W)) \to \mathbb{R} : \Gamma \mapsto \max\left\{0, 1 - \frac{1}{\varepsilon}\min_{g\in\Gamma} d(g, B_\varepsilon(g_0))\right\}.$$

It is easy to see that $f_{g_0,\varepsilon}$ is continuous. Furthermore, the set

$$\mathcal{F} = \{f_{g_0,\varepsilon} \mid g_0 \in G, \varepsilon > 0\}$$

contains the constant function equal to 1 and separates points. Due to the Stone-Weierstrass Theorem, $\mathcal{F}$ is dense in $C(\Omega(\curlywedge(W)))$.

Now, given $f \in \mathcal{F}$, Lemma 6.6 yields

$$\lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \int_{A_n} f(\Gamma - s)\, \mathrm{d}\Theta_G(s) = \lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \int_{A_n} f(\Gamma' - s)\, \mathrm{d}\Theta_G(s) \tag{6.1.2}$$

for all $\Gamma \in \Omega_i^f$ and $\Gamma' \in \beta^{-1}(\beta(\Gamma))$. Observe that (6.1.2) straightforwardly extends to all $f \in \mathcal{F}$ and hence to all $f \in \mathrm{cl}(\mathcal{F}) = C(\Omega(\curlywedge(W)))$. This shows that (6.1.1) holds for all $\Gamma \in \beta^{-1}(\beta(\Omega_i^f))$ with $f \in C(\Omega(\curlywedge(W)))$.

Since β sends μ_1 and μ_2 to the unique invariant measure $\Theta_{\mathbb{T}}$ on $\mathbb{T}$, we clearly have $\beta^{-1}(\beta(\Omega_1^f)) \cap \beta^{-1}(\beta(\Omega_2^f)) \neq \emptyset$. Hence,

$$\int_{\Omega(\curlywedge(W))} f\, \mathrm{d}\mu_1 = \lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \int_{A_n} f(\Gamma - s)\, \mathrm{d}\Theta_G(s) = \int_{\Omega(\curlywedge(W))} f\, \mathrm{d}\mu_2$$

for all $\Gamma \in \beta^{-1}(\beta(\Omega_1^f) \cap \beta(\Omega_2^f))$. Since $f \in C(\Omega(\curlywedge(W)))$ was chosen arbitrarily, we obtain $\mu_1 = \mu_2$. This finishes the proof. $\square$

6.2 Construction of a self similar Boundary

Given a planar CPS, we will use methods of the theory of minimal rotations to construct irregular proper windows W and V such that W is self similar and V has locally disjoint complements. To that end, we will construct a self similar and irredundant Cantor set with positive measure which serves as the boundary of W and V, respectively. By filling the gaps of C in certain ways - one preserving the self similarity of C, while the other destroys this property - we will obtain the desired windows (compare Chapter 5 for similar constructions). Before we start, we will discuss some basic observations regarding planar CPS (compare also Remark 4.20).

We consider a planar CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$. By Lemma 4.18 there exists an irrational matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$$

such that $A(\mathbb{Z}^2) = \mathcal{L}$. Without loss of generality we may put $d = 1$. This leads to

$$L^* = \pi_{\mathbb{R}}(\mathcal{L}) = \{nc + m \mid (n, m) \in \mathbb{Z}^2\} = \pi^{-1}(\{nc \mod 1 \mid n \in \mathbb{Z}\}),$$

where $\pi : \mathbb{R} \to \mathbb{S}$ denotes the canonical projection (compare also Chapter 11 for further discussion).

As pointed out in the previous section, the entropy of the Delone dynamical system $(\Omega(\curlywedge(W)), \mathbb{R})$ is related to the local structure of $W + h$ at points in $L^* \cap \partial W + h$. Given $h \in \mathbb{R}$, if $W \subseteq [0, 1]$, then a point in $L^* \cap \partial W + h$ corresponds to some $n \in \mathbb{Z}$ with $nc - h \mod 1 \in \partial W$. Thus, a self similar window $W \subseteq [0, 1]$ for the CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ can be understood as a subset $W \subseteq \mathbb{S}$ such that for all orbits $\mathcal{O}(x) = x + c\mathbb{Z}$ there are finitely many $n_1, \ldots, n_N \in \mathbb{Z}$ such that for all $y = x + nc \in \partial W \cap \mathcal{O}(x)$ there is $i = 1, \ldots, N$ and $\varepsilon > 0$ with

$$(B_\varepsilon(y) \cap W) + (n_i - n)c = B_\varepsilon(x + n_i c) \cap W.$$

Consistently with the terms introduced in Section 6.1 we will call such a subset of $\mathbb{S}$ *self similar*.

For simplicity, throughout this section we will work on the additive group $\mathbb{S}$ and omit writing $\mod 1$. By $d(x, 0)$ we denote the distance of $x \in \mathbb{R}$ to the nearest integer. Since we later have to distinguish between "left" and "right" on $\mathbb{S}$, we fix an orientation on $\mathbb{S}$.

Let $c \in [0, 1] \setminus \mathbb{Q}$. By R_c, we denote the rotation by c on $\mathbb{S}$, i.e. $R_c(x) = x + c$. Without loss of generality we may assume $|c| < \frac{1}{2}$. Let

$$q_1 = \min\{l \in \mathbb{N} \mid d(R_c^l(0), 0) < |c|\}$$

and define a sequence $(q_n)_{n\in\mathbb{N}}$ via

$$q_{n+1} = \min\{l \in \mathbb{N} \mid d(R_c^l(0), 0) < d(R_c^{q_n}(0), 0)\}.$$

Further, let I_n be the closed interval on $\mathbb{S}$ with endpoints 0 and $R^{q_n}(0)$. Then its length is given by $|I_n| = d(R^{q_n}(0), 0)$.

Proposition 6.9 ([DMVS12, Świ58]). *Suppose R_c is an irrational rotation on $\mathbb{S}$. Let*

$$\mathcal{P}_n = \{R_c^j(I_n) \mid 1 \leq j \leq q_{n+1}\} \cup \{R_c^j(I_{n+1}) \mid 1 \leq j \leq q_n\},$$

where q_n and I_n are defined as above. Then the following holds.

(i) $\mathbb{S} = \bigcup_{J\in\mathcal{P}_n} J$ and $\operatorname{int}(J_1) \cap \operatorname{int}(J_2) = \emptyset$ for each $J_1 \neq J_2 \in \mathcal{P}_n$.

(ii) For each $J \in \mathcal{P}_n$ and each $m > n$, there is $\mathcal{Q}_{J,m} \subseteq \mathcal{P}_m$ such that $J = \bigcup_{K\in\mathcal{Q}_{J,m}} K$.

(iii) If $J, J' \in \mathcal{P}_n$ and $J = R_c^l(J')$ for some $l \in \mathbb{N}$, then $\mathcal{Q}_{J,m} = R_c^l(\mathcal{Q}_{J',m}) = \{R_c^l(K) \mid K \in Q_{J',m}\}$ for all $m > n$.

Remark 6.10. (i) Note that (i) states that the elements of $\mathcal{P}_n$ basically partition $\mathbb{S}$ (compare also Remark 3.29) and (ii) yields that the partition of by elements of $\mathcal{P}_{n+1}$ is a refinement of that given $\mathcal{P}_n$. Point (iii) is to be understood as a self similarity of the respective partitions.

(ii) For $n = 1$, we provide this rough geometrical interpretation of the above proposition: The construction above yields two intervals I_1 and I_2 whose boundaries are contained in $\mathcal{O}(0)$. Applying R_c to I_1 transports its endpoints along $\mathcal{O}(0)$ such that the q_2 iterates of I_1 cover $\mathbb{S}$ up to q_1 small gaps of length $|I_2|$. Those gaps are filled by applying R_c to I_2. Note also that, for arbitrary $n \in \mathbb{N}$, the intervals in $\mathcal{P}_n$ are nested around 0 and we have $0 \in \partial I$ for all $I \in \mathcal{P}_n$.

Before we construct the boundary C, we want to give a rough outline of the constructions idea. We will construct C as the limit of a nested sequence $(C_l)_{l\in\mathbb{N}}$ of recursively defined compact subsets of $\mathbb{S}$. At each step l of the construction, the set C_l is obtained by removing elements of P_{n_l} from C_{l-1} (where $(n_l)_{l\in\mathbb{N}}$ is an appropriately chosen increasing sequence) so that by Proposition 6.9(ii), C_l is a union of intervals from $\mathcal{P}_{n_l}$. To establish the self similarity of the limit set C, we treat the intervals which comprise C_{l-1} equally. That is, roughly speaking, if $J_1, J_2 \in \mathcal{P}_{n_{l-1}}$ with $J_1, J_2 \subseteq C_{l-1}$ are translated copies of each other, say $R_c^n(J_1) = J_2$, then we keep $J \in \mathcal{Q}_{J_1,n_l}$ in C_l if and only if $R_c^n(J) \in \mathcal{Q}_{J_2,n_l}$ is kept in C_l.

Fix some $\varepsilon \in (0, 1)$. Let $(\beta_l)_{l\in\mathbb{N}}$ be a sequence of positive numbers such that

$$\sum_{l=1}^{\infty} 3\beta_l < \varepsilon$$

and let $(n_l)_{l\in\mathbb{N}}$ be a sequence of positive integers such that

$$\frac{|I_{n_l+1}|}{|I_{n_{l+1}}|} > \frac{1}{\beta_l}.$$

For technical reasons, without loss of generality we may assume $n_{l+1} \geq n_l + 6$ for all $l \in \mathbb{N}$. Note that this yields $\sharp\mathcal{Q}_{J,n_{l+1}} \geq 8$ for each $l \in \mathbb{N}$ and $J \in \mathcal{P}_{n_l}$.

We will recursively define a decreasing sequence of compact sets $C_l \subseteq \mathbb{S}$ (i.e., $C_l \subseteq C_{l-1}$) whose limit will be a Cantor set C satisfying the self similarity condition. Now suppose we have already constructed $C_l \subseteq C_{l-1} \subseteq \ldots C_1 = \mathbb{S}$. Given a set C_l, we call a connected component $J \subseteq C_l^c$ a *gap of level* k, $k \in \{2, \ldots, l\}$, if $J \cap C_k^c \neq \emptyset$ and $J \cap C_{k-1}^c = \emptyset$. We say an interval $J \in \mathcal{P}_{n_l}$ with $J \subseteq C_l$ is *k-accessible from the left/right* if its left/right endpoint is at the boundary of a gap of C_l of level k. We would like to point out that we will construct each C_l in such a way that each $J \in \mathcal{P}_{n_l}$ is accessible from at most one side.

Put $C_1 = \mathbb{S}$. Given C_l, we obtain C_{l+1} in the following way:

(C1) If $J \in \mathcal{P}_{n_l}$ is k-accessible from the left/right and $l-k$ is even, then remove from C_l the interior of the two left-most/right-most intervals and the interior of the right-most/left-most interval of $\mathcal{Q}_{J,n_{l+1}}$.

(C2) For all remaining $J \in \mathcal{P}_{n_l}$, remove from C_l the interior of the left-most and right-most interval of $\mathcal{Q}_{J,n_{l+1}}$.

(C3) Remove all isolated points from C_l which remain after applying (C1) and (C2).

Put $C = \bigcap_{l\in\mathbb{N}} C_l$. As an immediate consequence of the construction we obtain

Corollary 6.11. *We have $\mathcal{O}^+(0) \cap C = \emptyset$.*

Lemma 6.12. *The set C constructed above is a Cantor set with positive measure.*

Proof. As $(C_l)_{l\in\mathbb{N}}$ is a decreasing sequence of non-empty and compact sets, it follows immediately that C is non-empty and compact. By construction, C is perfect. Due to minimality of R_c and Corollary 6.11 the forward orbit $\mathcal{O}^+(0) = \{R_c^n(0) \mid n \in \mathbb{N}\}$ is a dense subset of C^c. Hence, C^c has dense interior and thus C is nowhere dense. Finally, observe that

$$\operatorname{Leb}(C) = \lim_{l\to\infty} \operatorname{Leb}(C_l) \geq 1 - \sum_{l=1}^{\infty} 3\beta_l > 1 - \varepsilon$$

which concludes the proof. □

To ensure self similarity of C, we introduce the following relation. Basically, along a common orbit, two points are related if the local structures (in terms of the elements of $\mathcal{P}_{n_l}$ in the l-th construction step) around those points are the same. Given $x \in \mathbb{S}$, $n \in \mathbb{Z}$ and $l \in \mathbb{N}$, we write $x \sim_l R_c^n(x)$ if

(i) $x, R_c^n(x) \notin \bigcup_{j=1}^{|n|} R_c^j(I_{n_l} \cup I_{n_l+1})$

(ii) for fixed $i \in \{0,1\}$ there exist $j_0, j_1 \in \{1, \ldots, q_{n_l+1-i}\}$ such that

 (a) $x \in \operatorname{int}(R_c^{j_0}(I_{n_l+i}))$ and $R_c^n(x) \in \operatorname{int}(R_c^{j_1}(I_{n_l+i}))$,

 (b) $R_c^{j_0}(I_{n_l+i})$ and $R_c^{j_1}(I_{n_l+i})$ are

  - from the same side k- and k'-accessible, respectively, with $k-k'$ even

 or

  - are not accessible at all.

Recalling the proof of Lemma 6.12, we have $\mathcal{O}^+(0) \cap C = \emptyset$. Thus, for each $x \in C$, we have that $x \in R_c^{j_0}(I_{n_l+i})$ actually means that $x \in \operatorname{int}(R_c^{j_0}(I_{n_l+i}))$.

In particular, if points are in relation for some $l \in \mathbb{N}$, then this relation will be preserved under succeeding construction steps (C1) - (C3), i.e. if we have $x \sim_l R_c^n(x)$, then $x \sim_k R_c^n(x)$ for all $k > l$.

Lemma 6.13. *Consider $x \in C$. If $x \sim_l R_c^n(x)$ for some $l \in \mathbb{N}$ and $n \in \mathbb{Z}$ with $R_c^n(x) \in C_l$, then $R_c^n(x) \in C$.*

Proof. Since $x \sim_l R_c^n(x)$, there exist $j_0, j_1 \in \mathbb{N}$ which satisfy conditions (i) and (ii) above. Due to (i) and Proposition 6.9(i) we obtain $j_0 + n = j_1$. Hence, the distance of $R_c^n(x)$ to the left endpoint of $R_c^{j_1}(I_{n_l+i}) = R_c^{j_0+n}(I_{n_l+i})$ equals the distance of x the left endpoint of $R_c^{j_0}(I_{n_l+i})$. Note that the same statement holds for the right endpoints. By Proposition 6.9(iii) we further have

$$\mathcal{Q}_{R_c^{j_1}(I_{n_l+i}),n_{l+1}} = R_c^n\left(\mathcal{Q}_{R_c^{j_0}(I_{n_l+i}),n_{l+1}}\right).$$

Then, by definition of C_{l+1}, we obtain $R_c^n(x) \in C_{l+1}$ and $x \sim_{l+1} R_c^n(x)$. Induction on l then yields $R_c^n(x) \in C$. □

On the other hand, if points along one orbit are contained in the resulting Cantor set, both points have to be in relation at some point during the construction of C.

Lemma 6.14. *Assume $x \in C$ and $y \in \mathcal{O}(x) \cap C$. Then we have $x \sim_l y$ for sufficiently large $l \in \mathbb{N}$.*

Proof. Without loss of generality we may assume $y = R_c^{-n}(x)$ for some $n \in \mathbb{N}$. By Proposition 6.9(i) there exists some $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ there is $i_l \in \{0,1\}$ such that $x \in \bigcup_{j=1}^{q_{n_l+1-i_l}} R_c^j(I_{n_l+i_l})$. Together with Corollary 6.11 this yields

$$x \in \operatorname{int}\left(\bigcup_{j=2n+1}^{q_{n_l+1-i_l}} R_c^j(I_{n_l+i_l})\right).$$

Hence,

$$y \in \operatorname{int}\left(\bigcup_{j=n+1}^{q_{n_l+1-i_l}} R_c^j(I_{n_l+i_l})\right).$$

This means, there are $j_0^l, j_1^l \in \{1, \dots, q_{n_l+1-i_l}\}$ with

$$x \in \operatorname{int}\left(R_c^{j_0^l}(I_{n_l+i_l})\right) \text{ and } y \in \operatorname{int}\left(R_c^{j_1^l}(I_{n_l+i_l})\right) = \operatorname{int}\left(R_c^{j_0^l-n}(I_{n_l+i_l})\right).$$

Recalling the proof of Lemma 6.13, we see that x and y have the same distance to the endpoints of $R_c^{j_0^l}(I_{n_l+i_l})$ and $R_c^{j_1^l}(I_{n_l+i_l})$, respectively, and that

$$\mathcal{Q}_{R_c^{j_1^l}(I_{n_l+i_l}),n_{l+1}} = R_c^{-n}\left(\mathcal{Q}_{R_c^{j_0^l}(I_{n_l+i_l}),n_{l+1}}\right).$$

It remains to show (ii)(b) for sufficiently large $l \in \mathbb{N}$. To that end, fix some $l \geq l_0$. Then we have to consider the following cases.

(i) *$R_c^{j_0^l}(I_{n_l+i_l})$ and $R_c^{j_1^l}(I_{n_l+i_l})$ are accessible from different sides.* By construction of C_{l+1} and $\sharp\mathcal{Q}_{R_c^{j_0}(I_{n_l+i_0}),n_l+1} \geq 6$, we have that either

$$R_c^{j_0^{l+1}}(I_{n_{l+1}+i_{l+1}}) \text{ and } R_c^{j_1^{l+1}}(I_{n_{l+1}+i_{l+1}}) \text{ are accessible from the same side}$$

or at least one of the two intervals is not accessible at all. Hence, we have reduced the problem to one of the following cases.

(ii) *$R_c^{j_0^l}(I_{n_l+i_l})$ and $R_c^{j_1^l}(I_{n_l+i_l})$ are k- and k'-accessible from the same side and $k - k'$ is odd.* We may assume without loss of generality that both intervals are accessible from the right and that $l - k$ is even. If $R_c^{j_0^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is still accessible from the right, then $R_c^{j_1^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is not accessible anymore, since $R_c^{j_1^l}(I_{n_l+i_l})$ has been dealt with according to (C2) while $R_c^{j_0^l}(I_{n_l+i_l})$ has been dealt with according to (C1). Hence, we are in case (iv). If $R_c^{j_0^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is not accessible from the right anymore, then the same is true for $R_c^{j_1^{l+1}}(I_{n_{l+1}+i_{l+1}})$ and hence either both are $l+1$-accessible from the left or not accessible at all. In both cases we are done.

(iii) *$R_c^{j_0^l}(I_{n_l+i_l})$ is k-accessible from some side with $l-k$ even, while $R_c^{j_1^l}(I_{n_l+i_l})$ is not accessible.* Without loss of generality assume that $R_c^{j_0^l}(I_{n_l+i_l})$ is accessible from the right. If $R_c^{j_0^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is accessible from the left or not accessible at all, the same holds for $R_c^{j_1^{l+1}}(I_{n_{l+1}+i_{l+1}})$ and we are done. If $R_c^{j_0^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is still k-accessible from the right, then $R_c^{j_1^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is not accessible which leads to case (iv).

(iv) $R_c^{j_0^l}(I_{n_l+i_l})$ *is* k*-accessible from some side with* $l-k$ *odd, while* $R_c^{j_1^l}(I_{n_l+i_l})$ *is not accessible.* Without loss of generality, we assume that $R_c^{j_0^l}(I_{n_l+i_l})$ is accessible from the right. Note that $R_c^{j_0^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is still k-accessible from the right if and only if $R_c^{j_1^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is k'-accessible from the right as well with $k' = l+1$. Since $k'-k$ is even we are done. In case $R_c^{j_0^{l+1}}(I_{n_{l+1}+i_{l+1}})$ is accessible from the left or not accessible at all, the same holds true for $R_c^{j_1^{l+1}}(I_{n_{l+1}+i_{l+1}})$.

Thus, the proof is complete. □

Now we are ready to proof that C is indeed self similar.

Proposition 6.15. *Let* $x, y \in C$ *such that* $y = R_c^n(x)$ *for some* $n \in \mathbb{Z}$*. Then there exists* $\varepsilon > 0$ *such that*

$$R_c^n(B_\varepsilon(x) \cap C) = B_\varepsilon(y) \cap C.$$

Proof. Due to Lemma 6.14 there exists $l \in \mathbb{N}$ such that $x \sim_l y$. Thus, there are $i \in \{0,1\}$ and $j_0, j_1 \in \{1, \ldots, q_{n_l+1-i}\}$ such that

$$x \in \operatorname{int}(R_c^{j_0}(I_{n_l+i})) \subseteq C_l \text{ and } R_c^n(x) \in \operatorname{int}(R_c^{j_1}(I_{n_l+i})) \subseteq C_l.$$

Choose $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq R_c^{j_0}(I_{n_l+i})$. By the assumptions, we then also have $B_\varepsilon(y) \subseteq R_c^{j_1}(I_{n_l+i})$. Suppose there exists some $z \in B_\varepsilon(x) \cap C$. Then we obtain $R_c^n(z) \in C_l$ and $z \sim_l R_c^n(z)$. By Lemma 6.13, we then have $R_c^n(z) \in C$. But this means $R_c^n(B_\varepsilon(x) \cap C) \subseteq B_\varepsilon(y) \cap C$. By analogous arguments we obtain the opposite inclusion. □

6.3 Filling the Gaps of the self similar Boundary

Now we are going to construct the windows W and V mentioned in the previous section. Both windows will satisfy $\partial W = \partial V = C$ and will give rise to Delone dynamical systems which are almost automorphic extension of $(\mathbb{S}, R_c)$. Further, depending on the windows the fibres of the corresponding factor map will have different properties:

- The fibres of $\beta : \Omega(\curlywedge(W)) \to \mathbb{T}$ will contain at most two elements (as we will see, they will contain almost surely exactly two elements). This allows $\Omega(\curlywedge(W))$ to carry two distinct ergodic measures.
- In contrast, the fibres of $\beta : \Omega(\curlywedge(V)) \to \mathbb{T}$ will have infinite cardinality, whereas $\Omega(\curlywedge(V))$ will be uniquely ergodic.

Remark 6.16. Regarding the second statement, it turns out that infinite fibres are a necessary requirement for an irregular almost automorphic system to be *mean equicontinuous* ([FGL18]). We refer to [DG16] for a discussion of mean equicontinuity.

Similar to the constructions in Section 5.4 and 5.5, our goal is to fill the gaps of C in such a way that the resulting set becomes a proper window. As a preparation, we first take a closer look at the accessible points of C. To that end, let us provide the following observation.

Proposition 6.17. *For all* $l \in \mathbb{N}$ *there are* $G^1_{l+1}, G^2_{l+1} \in \mathcal{P}_{n_{l+1}}$ *such that for each* $J \in \mathcal{P}_{n_l}$ *the left-most interval (the interval second from left) of* $\mathcal{Q}_{G,n_{l+1}}$ *is a translated copy of* G^1_{l+1} (G^2_{l+1})*.*

A similar statement holds if we replace left by right.

Proof. We only consider the "left case". Without loss of generality, we may assume that I_{n_l+2} is an interval to the right of zero (otherwise, we may proceed with n_l+3 instead of n_l+2). Now, recall that $q_{n+1} \geq q_n + q_{n+1}$ for each $n \in \mathbb{N}$ (in fact, if a_n is the n-th coefficient of the continued fraction expansion of c, then $q_{n+1} = a_n q_n + q_{n+1}$, [Kur03]). Given any $G \in \mathcal{P}_{n_l}$, this yields that the left-most interval of $\mathcal{Q}_{G,n_l+2}$ is a translated copy of I_{n_l+2}. Since we assume $n_{l+1} \geq n_l + 6$, the statement follows by means of Proposition 6.9(iii). □

Let $(G_n)_{n\in\mathbb{N}}$ be a labeling of the gaps of C and denote by $x_n \in C$ the right endpoint of G_n. We say that G_n is of *level* l if $G_n \cap C_l^c \neq \emptyset$ and $G_n \cap C_{l-1}^c = \emptyset$. Assume G_n is of level l. Let y_n denote the isolated point in G_n which had to be removed in step (C3) of the construction of C_l. Let k denote the level of G_n and k' the level of $G_{n'}$ and assume without loss of generality that $k < k'$.

Suppose $k' - k$ is even, i.e. both gaps are of odd level or both gaps are of even level. Then we have

$$x_n - x_{n'} = y_n - y_{n'} + \sum_{l=k}^{k'-1} \alpha_{-1^{l-k};l},$$

where $\alpha_{1;l} = |G_l^1|$ and $\alpha_{-1;l} = |G_l^1| + |G_l^2|$ (recall, that every second step of the construction of $C_{k'}$, we remove two intervals on either side of $G_n \cap C_k$. Hence, $x_n - x_{n'}$ is an integer multiple of c. This means, all right endpoints of the even-level gaps of C belong to one orbit and all right endpoints of odd-level gaps belong to one orbit.

Now suppose $k' - k$ is odd, i.e. one gap is of odd level and one gap is of even level. Then we have

$$x_n - x_{n'} = y_n - y_{n'} + \sum_{l=k}^{k'-1} \alpha_{-1^{l-k};l} + \sum_{l=k'}^{\infty} (-1)^{l-k'} |G_l^2|.$$

Note that, by possibly going over to a subsequence, we may assume that $2\sum_{l=k+1}^{\infty} |G_l^2| < |G_k^2|$ for all integers $k \geq 2$. Hence, $\sum_{l=2}^{\infty}(-1)^{l_j}|G_{l_j}^2| \neq \sum_{l=2}^{\infty}(-1)^{l'_j}|G_{l'_j}^2|$ for distinct subsequences (n_{l_j}) and $(n_{l'_j})$ of (n_l). Since there are clearly uncountably many subsequences but only countable many integer multiples of c, we may assume without loss of generality that $\sum_{l=2}^{\infty}(-1)^l|G_l^2|$ (and thus also $\sum_{l=k'}^{\infty}(-1)^{l-k'}|G_l^2|$) is not an integer multiple of c. Then x_n and $x_{n'}$ belong to different orbits of R_c.

Thus we have proven

Lemma 6.18. *Let $(G_n)_{n\in\mathbb{N}}$ be a labeling of the gaps of C and x_n denote the right endpoints of G_n. Then there exist points $x, y \in \mathbb{S}$ such that the following holds.*

(i) If n is even, then $x_n \in \mathcal{O}(x)$.

(ii) If n is odd, then $x_n \in \mathcal{O}(y)$.

Further, $\mathcal{O}(x) \cap \mathcal{O}(y) = \emptyset$.

A similar statement holds for the left endpoints, i.e. the left endpoints of even-level gaps belong to one orbit and the left endpoints of odd-level gaps belong to a different one.

Observe that two gaps of C are of equal length if and only if they are of the same level. This immediately shows

Lemma 6.19. *The Cantor set C is irredundant.*

Without loss of generality, we may assume in the following that G_{2n} is of an even level while G_{2n+1} is of an odd level for each $n \in \mathbb{N}$. Then define the window W by

$$W = C \cup \bigcup_{n\in\mathbb{N}} G_{2n}.$$

Clearly, the boundary of W is given by the Cantor set C. By Lemma 6.19, we obtain irredundancy of W. Furthermore, we observe that between two gaps of level l there always exists a gap of level $l + 1$. Thus, we obtain

Lemma 6.20. *We have $\partial W = C$ and $W = \mathrm{cl}(\mathrm{int}(W))$. Further, W is irredundant.*

Lemma 6.21. *Suppose we are given $x, y \in C$ with $y = R_c^n(x)$ for some $n \in \mathbb{Z}$. Then there exists $\varepsilon > 0$ such that*

$$R_c^n\left(B_\varepsilon(x) \cap W\right) = B_\varepsilon(y) \cap W.$$

Proof. By Proposition 6.15 there exists $\varepsilon > 0$ such that $R_c^n\,(B_\varepsilon(x) \cap C) = B_\varepsilon(y) \cap C$. In particular, each left/right endpoint $x' \in B_\varepsilon(x)$ of a gap G_n which intersects $B_\varepsilon(x)$ corresponds to a left/right endpoint $y' = R_c^n(x') \in B_\varepsilon(y)$ of a gap $G_{n'}$ which intersects $B_\varepsilon(y)$. As J_n and $J_{n'}$ have endpoints of one and the same orbit (compare Lemma 6.18), the above discussion shows that, by definition of W, that $G_n \subseteq W$ if and only if $G_{n'} \subseteq W$. Hence, $R_c^n(B_\varepsilon(x) \cap W) = B_\varepsilon(y) \cap W$. □

Recalling the discussion at the beginning of the previous section, this window W is self similar in the sense of the definition of Section 6.1.

Next, we turn to the construction of the window V. Let $(G_n)_{n\in\mathbb{N}}$ be a labeling of the gaps of C. Given a gap G_n and a level $k \geq 2$, let $G(k, G_n)$ be a gap of level k which minimizes the distance to G_n. Set

$$V = \mathbb{S} \setminus \bigcup_{k\geq 2} G(k, G_k).$$

By construction, V contains exactly one gap of each level.

Lemma 6.22. *The set V is proper, irredundant and we have $\partial V = C$. Further, V has locally disjoint complements.*

Proof. By construction, the first three properties are satisfied. Let $x, R_c^n(x) \in C$ for some $n \in \mathbb{Z}$. By self similarity of C (compare Proposition 6.15) there exists some $\varepsilon > 0$ such that

$$R_c^n(B_\varepsilon(x) \cap C) = B_\varepsilon(R_c^n(x)) \cap C.$$

Note that V^c is a disjoint union of gaps which all belong to different levels. Thus we obtain

$$R_c^n(B_\varepsilon(x) \cap V^c) \cap B_\varepsilon(R_c^n(x)) \cap V^c = \emptyset.$$

Hence, V has locally disjoint complements. □

Concluding this section we want to point out an immediate consequence of the discussions above and in Chapter 5.

Proposition 6.23. *Let $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ be a planar CPS. Equip $\mathcal{W} = \{W \subseteq \mathbb{R} \mid W$ is a proper window$\}$ with the Hausdorff metric. Then the map*

$$\mathcal{W} \to \mathbb{R} : W \mapsto h_{\mathrm{top}}(\Omega(人(W)), \mathbb{R})$$

is neither upper nor lower semicontinuous.

Proof. Suppose C as well as the gaps G_n are constructed as above. Let $\Sigma^+ = \{0,1\}^{\mathbb{N}}$. By $\mathbb{P}$ we denote the Bernoulli measure on Σ^+ (compare Section 5.4 for further discussion). By Theorem 5.18, the window

$$W(\omega) = C \cup \bigcup_{n\in\mathbb{N}:\omega_n=1} G_n$$

yields a Delone dynamical system such that

$$h_{\mathrm{top}}(\Omega(人(W(\omega))), \mathbb{R}) > 0$$

for $\mathbb{P}$-almost every $\omega \in \Sigma^+$.

Let W_0 denote the self similar window constructed in this section (compare Lemmas 6.20 and 6.21). Given two sequences $\sigma, \omega \in \Sigma^+$, we denote by $z(n; \sigma, \omega) \in \Sigma^+$ the sequence which coincides with σ on the first n entries and with ω on all of the remaining entries. Now let ω be chosen such that $W(\omega)$ is proper (compare Lemma 5.15) and yields a dynamical hull with positive topological entropy, while σ is chosen such that $W(\sigma) = W_0$. Then for each $n \in \mathbb{N}$ we have that

$$W(z(n; \sigma, \omega)) \text{ and } W(z(n; \omega, \sigma))$$

are proper. Furthermore, we have

$$h_{\mathrm{top}}(\Omega(人(W(z(n; \sigma, \omega))), \mathbb{R}) = h_{\mathrm{top}}(\Omega(人(W(\omega))), \mathbb{R}) > 0$$

as well as $h_{\mathrm{top}}(\Omega(人(W(z(n; \omega, \sigma)))), \mathbb{R}) = 0$. Thus, the statement holds. □

6.4 Higher Dimensional Euclidean CPS and Topological Entropy

Our goal of this section is to provide a whole class of higher dimensional CPS with irregular windows whose associated Delone dynamical systems have zero entropy. To that end, consider an Euclidean CPS $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ with proper window $W \subseteq \mathbb{R}$. To this given CPS we will associate N planar CPS, whose respective lattices may be constructed from the given lattice $\mathcal{L}$. We are going to show the following: if one of the planar CPS yields a dynamical hull with vanishing entropy, then the dynamical hull corresponding to the given N-dimensional CPS has also zero entropy.

As discussed in Lemma 4.18 and Remark 4.20 (see also Chapter 11), we may represent the lattice as $\mathcal{L} = A(\mathbb{Z}^{N+1})$, where $A = (a_{ij}) \in \mathrm{GL}(N+1, \mathbb{R})$ is an irrational matrix. Let $v_i = (a_{1i}, \ldots, a_{Ni})^T$ denote the first N entries of the columns of A and put $c_i = a_{N+1,i}$. Without loss of generality we may assume $c_{N+1} = 1$ as well as $W \subseteq [0,1]$. Additionally, we assume without loss of generality that $\{c_1, \ldots, c_{N+1}\}$ is a set of rationally independent values. Then we have

$$L^* = \pi_{N+1}(\mathcal{L}) = \left\{ \sum_{i=1}^{N} n_i c_i + n_{N+1} : n_i \in \mathbb{Z} \right\} = \pi^{-1}\left(\left\{ \sum_{i=1}^{N} n_i c_i \mod 1 : n_i \in \mathbb{Z} \right\} \right),$$

where $\pi : \mathbb{R} \to \mathbb{S}$ denotes the canonical projection and $\pi_{N+1} : \mathbb{R}^{N+1} \to \mathbb{R}$ the canonical projection to the internal space. Hence, L^* is the lift of an orbit of a $\mathbb{Z}^N$-rotation on $\mathbb{S}$ with N rationally independent rotation numbers c_i. To each rotation number we may associate a set $L_i^* = \pi^{-1}(\{nc_i \mod 1 \mid n \in \mathbb{Z}\})$.

For $i = 1, \ldots, N$ put

$$A_i = \begin{pmatrix} a_{ii} & a_{i,N+1} \\ c_i & 1 \end{pmatrix}.$$

Observe that each A_i is a regular matrix with rationally independent rows. Thus, $\mathcal{L}_i = A(\mathbb{Z}^2)$ is an irrational lattice in $\mathbb{R}^2$. In particular we have $\pi_2(\mathcal{L}_i) = L_i^*$. In this way, we associate N planar CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L}_i)$ with window $W \subseteq \mathbb{R}$ to a given CPS $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ with exactly the same window $W \subseteq \mathbb{R}$. We denote the corresponding Delone dynamical systems by $(\Omega(\curlywedge_i(W)), \varphi_i)$.

It is easy to see that the following equivalences hold for each $n \in \mathbb{Z}$ and $i = 1, \ldots, N$:

$$\text{(6.4.1)} \quad nc_i \mod 1 \in W \iff nv_i - \lfloor nc_i \rfloor v_{N+1} \in \curlywedge(W) \iff na_{ii} - \lfloor nc_i \rfloor a_{i,N+1} \in \curlywedge_i(W).$$

Now fix some $t \in \mathbb{R}$. Given a point $p = nv_1 + \sum_{i=2}^N m_i v_i + kv_{N+1} \in \curlywedge(W+t)$, put $\boldsymbol{m}_p = (m_2, \ldots, m_N) \in \mathbb{Z}^{N-1}$. Note that $nv_1 + \sum_{i=2}^N m_i v_i + kv_{N+1} \in \curlywedge(W+t)$ is equivalent to $nc_1 + k \in W + t - \sum_{i=2}^N m_i c_i$. For given $\boldsymbol{m} \in \mathbb{Z}^{N-1}$, we define the *pseudoline*

$$G_{W+t}(\boldsymbol{m}) = \sum_{i=2}^N m_i v_i + \left\{ nv_1 + kv_{N+1} : n, k \in \mathbb{Z}, nc_1 + k \in W + t - \sum_{i=2}^N m_i c_i \right\} \subseteq \curlywedge(W+t).$$

Pseudolines have the following properties:

(i) The set $\{G_{W+t}(\boldsymbol{m}) \mid \boldsymbol{m} \in \mathbb{Z}^{N+1}\}$ partitions $\curlywedge(W+t)$.

(ii) Rationally independence of a_{11} and $a_{1,N+1}$ ensure that the restriction of π_1 to $G_{W+t}(\boldsymbol{m})$ is injective for all $\boldsymbol{m} \in \mathbb{Z}^{N-1}$.

(iii) For all $p \in \curlywedge(W+t)$ we have

$$\text{(6.4.2)} \qquad \pi_1(G_{W+t}(\boldsymbol{m}_p) = \sum_{i=2}^N m_i a_{1i} + \curlywedge_1\left(W + t - \sum_{i=2}^N m_i c_i \right) \in \Omega(\curlywedge_1(W+t)).$$

Now, for given $\boldsymbol{m} \in \mathbb{Z}^{N-1}$ we define the line

$$\mathcal{G}(\boldsymbol{m}) = \sum_{i=2}^{N} m_i v_i + \mathbb{R}\left(\frac{1}{c_1} v_1 - v_{N+1}\right) \subseteq \mathbb{R}^N.$$

Notice that there exists $C > 0$ independent of t such that for each $G_{W+t}(\boldsymbol{m})$ we have $G_{W+t}(\boldsymbol{m}) \subseteq B_C(\mathcal{G}(\boldsymbol{m}))$. Due to by regularity of A we have $\left(\frac{1}{c_1} v_1 - v_{N+1}\right) \notin \operatorname{span}\{v_2, \ldots, v_N\}$. Therefore, we immediately obtain the following statement.

Lemma 6.24. *Suppose $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ is a CPS with proper window $W \subseteq \mathbb{R}$. Then there exists $\kappa > 0$ such that for each $t \in \mathbb{R}$ we have*

$$\sharp\left\{G_{W+t}(\boldsymbol{m}) \mid \boldsymbol{m} \in \mathbb{Z}^{N-1}, G_{W+t}(\boldsymbol{m}) \cap B_M^N(0) \neq \emptyset\right\} \leq \kappa \cdot \operatorname{Leb}\left(B_M^{N-1}(0)\right),$$

where $B_M^d(0) \subseteq \mathbb{R}^d$ denotes the d-dimensional M-ball centered at 0.

Now we may state the main result of this section.

Proposition 6.25. *Let $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ be a CPS with proper window $W \subseteq \mathbb{R}$. Furthermore, assume that there exists $i \in \{1, \ldots, N\}$ such that $h_{\mathrm{top}}(\varphi_i) = 0$. Then we have $h_{\mathrm{top}}^{\xi}(\varphi) = 0$ for all $\xi \in \mathbb{T}$.*

Proof. Without loss of generality, we may assume that $W \subseteq [0,1]$ as well as $h_{\mathrm{top}}(\varphi_1) = 0$. We equip $\mathbb{R}$ as well as $\mathbb{R}^N$ with the Euclidean metric and consider the entropy of φ_1 and φ by averaging over the van Hove sequence given by one-dimensional balls $\left(B_M^1(0)\right)_{M \in \mathbb{N}}$ and N-dimensional balls $(B_M(0))_{M \in \mathbb{N}}$, respectively.

Fix some $\xi = [0,t]_{\mathcal{L}} \in \mathbb{T}^{N+1}$. Put

$$r = \frac{1}{2} \min\left\{\inf_{p \neq q \in \curlywedge(W+t)} \|p-q\|, \inf_{p \neq q \in \curlywedge_1(W)} \|p-q\|\right\}$$

and let $\varepsilon \in (0,r)$. Given $M \in \mathbb{N}$, let $S_1(\varepsilon, M)$ denote an (ε, M)-spanning set for $\Omega(\curlywedge(W))$ with minimal cardinality. Put $P_1(\varepsilon, M) = \sharp S_1(\varepsilon, M)$. Our goal is to construct an (ε, M) spanning set $S^{\xi}(\varepsilon, M)$ for $\beta^{-1}(\xi)$ which satisfies

$$\sharp S^{\xi}(\varepsilon, M) \leq P_1(\varepsilon, M)^{\kappa \operatorname{Leb}\left(B^{N-1}_{M+1/\varepsilon}(0)\right)}. \tag{6.4.3}$$

To that end, we want to point out that Equation 4.4.1 together with our choice of ε imply that two Delone sets $\Lambda, \Gamma \in \beta^{-1}(\xi)$ satisfy $\max_{s \in B_M(0)} d(\Lambda - s, \Gamma - s) < \varepsilon$ if we have $\Lambda \cap B_{1/\varepsilon}(0) = \Gamma \cap B_{1/\varepsilon}(0)$ for all $s \in B_M(0)$. Since $\curlywedge(W+t)$ can be covered by pseudolines, the above is the case if for all such s and $p \in B_{1/\varepsilon}(s) \cap \curlywedge(W+t)$ we have that

$$G_{W+t}(\boldsymbol{m}_p) \cap \Lambda \cap B_{1/\varepsilon}(s) = G_{W+t}(\boldsymbol{m}_p) \cap \Gamma \cap B_{1/\varepsilon}(s).$$

Since the restriction of π_1 to pseudolines is injective, this is equivalent to

$$\pi_1(G_{W+t}(\boldsymbol{m}_p) \cap \Lambda \cap B_{1/\varepsilon}(s)) = \pi_1(G_{W+t}(\boldsymbol{m}_p) \cap \Gamma \cap B_{1/\varepsilon}(s)). \tag{6.4.4}$$

Now, let $\Gamma \in \beta^{-1}(\xi)$. By Lemma 4.33 there is a sequence $(t_j)_{j \in \mathbb{N}} \subseteq L^*$ such that we have $\Gamma = \lim_{j \to \infty} \curlywedge(W + t_j)$. Let $\boldsymbol{m} \in \mathbb{Z}^{N-1}$. Observe that we have

$$\begin{aligned}\pi_1(G_{W+t}(\boldsymbol{m}) \cap \Gamma) &= \pi_1(G_{W+t}(\boldsymbol{m}) \cap \lim_{j \to \infty} \curlywedge(W + t_j)) \\ &= \lim_{j \to \infty} \pi_1(G_{W+t_j}(\boldsymbol{m}) \cap \curlywedge(W + t_j))\end{aligned}$$

which is an element of $\Omega(\curlywedge_1(W))$ due to 6.4.2. Hence, by definition of $S_1(\varepsilon, M)$, there is $\Delta \in S_1(\varepsilon, M)$ with

$$\max_{s \in B_M^1(0)} d(\pi_1(G_{W+t}(\boldsymbol{m}) \cap \Gamma) - s, \Delta - s) < \varepsilon.$$

In particular, we have

$$\pi_1\left(G_{W+t}(\boldsymbol{m}) \cap \Gamma \cap B_{M+1/\varepsilon}(0)\right) \subseteq \Delta + \delta \tag{6.4.5}$$

for some $\delta \in \mathbb{R}$ with $|\delta| < \varepsilon$. Since $\varepsilon < r$, we have that for fixed $\boldsymbol{m} \in \mathbb{Z}^{N-1}$ and $\Delta \in S_1(\varepsilon, M)$ there is at most one such δ for which 6.4.5 is satisfied for some $\Gamma \in \Omega(人(W+t))$. If 6.4.5 holds, we say Γ *realizes the local configuration of* Δ *along* $G_{W+t}(\boldsymbol{m})$. We define a relation $\sim$ on $\beta^{-1}(\xi)$ by putting $\Gamma \sim \Lambda$ if Γ and Λ realize the same local configuration along $G_{W+t}(\boldsymbol{m}_p)$, where $p \in B_{1/\varepsilon}(s) \cap 人(W+t)$. It is easy to see that $\sim$ is an equivalence relation on $\beta^{-1}(\xi)$. Thus, we obtain

$$max_{s \in B_M(0)} d(\Lambda - s, \Gamma - s) < \varepsilon \text{ if } \Gamma \sim \Lambda.$$

Finally, we set $S^\xi(\varepsilon, M)$ to be a set which contains one representative for each equivalence class of $\sim$. By our considerations above, this set is (ε, M)-spanning for $\beta^{-1}(\xi)$.

Note that there are at most $P_1(\varepsilon, M)$ possible configurations realized along each $G_{W+t}(\boldsymbol{m}_p) \cap B_{M+1/\varepsilon}(0)$. Furthermore, according to Lemma 6.24, the number of pseudolines intersecting $B_{M+1/\varepsilon}(0)$ is bounded by $\kappa \cdot \mathrm{Leb}\left(B^{N-1}_{M+1/\varepsilon}(0)\right)$. Hence, we obtain 6.4.3. Thus,

$$\begin{aligned}
h^\xi_{\mathrm{top}}(\varphi) &\le \lim_{\varepsilon \to 0} \limsup_{M \to \infty} \frac{\kappa \cdot \mathrm{Leb}\left(B^{N-1}_{M+1/\varepsilon}(0)\right)}{\mathrm{Leb}(B_M(0))} \log P_1(\varepsilon, M) \\
&= \lim_{\varepsilon \to 0} \limsup_{M \to \infty} \frac{2\kappa}{\sqrt{\pi}} \left(\frac{1}{M\varepsilon} + 1\right)^{N-1} \frac{\log P_1(\varepsilon, M)}{\mathrm{Leb}(B^1_M(0))} \\
&\le \frac{2\kappa}{\sqrt{\pi}} \lim_{\varepsilon \to 0} \limsup_{M \to \infty} \frac{\log P_1(\varepsilon, M)}{\mathrm{Leb}(B^1_M(0))} = 0.
\end{aligned}$$

This finishes the proof. □

6.5 Invariant Measures, Dynamical Spectrum and Diffraction

Consider a CPS $(G, H, \mathcal{L})$ with proper window $W \subseteq H$ and associated torus parametrization $\beta : (\Omega(人(W)), \varphi) \to (\mathbb{T}, \omega)$. We call a measurable map $\gamma : \mathbb{T} \to \Omega(人(W))$ an *invariant graph (for $\Omega(人(W))$)* if γ is a G-map and for all $s \in G$ and $h \in H$ holds

$$\beta(\gamma([s,h]_{\mathcal{L}})) = [s,h]_{\mathcal{L}}.$$

Given an invariant graph γ, we define the associated *graph measure* by setting

$$\mu_\gamma(A) = \Theta_{\mathbb{T}}(\gamma^{-1}(A))$$

for all measurable $A \subseteq \Omega(人(W))$. Observe that, since $\Theta_{\mathbb{T}}$ is ergodic, the graph measure is an ergodic measure on $(\Omega(人(W)), G)$. Now let

$$U_\gamma : L^2(\Omega(人(W)), \mu_\gamma) \to L^2(\mathbb{T}, \Theta_{\mathbb{T}}) : f \mapsto f \circ \gamma.$$

Then, for all $f, g \in L^2(\Omega(人(W)), \mu_\gamma)$, we have

$$\begin{aligned}
\langle U_\gamma f, U_\gamma g\rangle_{L^2(\mathbb{T},\Theta_{\mathbb{T}})} &= \int_{\mathbb{T}} \overline{U_\gamma f}\, U_\gamma g \,\mathrm{d}\Theta_{\mathbb{T}} = \int_{\mathbb{T}} (\overline{f}\, g) \circ \gamma \,\mathrm{d}\Theta_{\mathbb{T}} \\
&= \int_{\Omega(人(W))} \overline{f}\, g \,\mathrm{d}\mu_\gamma = \langle f, g\rangle_{L^2(\Omega(人(W)),\mu_\gamma)}.
\end{aligned}$$

Due to the definition of U_γ and γ being an invariant graph, for all $g \in L^2(\mathbb{T}, \Theta_{\mathbb{T}})$ we have $U_\gamma(g \circ \beta) = g$. Hence, U_γ is bijective and thus an isometric isomorphism.

Recalling the notions introduced in Section 3.5, for $s \in G$ and $f \in L^2(\Omega(人(W)), \mu_\gamma)$ we have

$$T^s(U_\gamma f)(\cdot) = f \circ \gamma(\omega_{-s}(\cdot)) = f(\varphi_{-s}(\gamma(\cdot))) = (T^s f) \circ \gamma(\cdot) = U_\gamma(T^s f)(\cdot).$$

Altogether, we have proven

Proposition 6.26. *Let $(G, H, \mathcal{L})$ be a CPS with proper window $W \subseteq H$ and $\gamma : \mathbb{T} \to \Omega(人(W))$ be an invariant graph with corresponding graph measure μ_γ. Then the dynamical hull $(\Omega(人(W)), G, \mu_\gamma)$ has pure point spectrum and all eigenfunctions are continuous.*

Now suppose for almost every $\xi = [s, h]_{\mathcal{L}} \in \mathbb{T}$ the fibre $\beta^{-1}(\xi)$ contains a unique maximal element $\Gamma_+ = \Gamma_+(\xi)$ or a unique minimal element $\Gamma_- = \Gamma_-(\xi)$ with respect to set inclusion. We set

$$\gamma_\pm : \mathbb{T} \to \Omega(人(W)) : \xi \mapsto \Gamma_\pm(\xi).$$

Proposition 6.27. *Suppose almost every fibre contains an element Γ_+ (Γ_-) as above. Then γ_+ (γ_-) is an invariant graph.*

Proof. As the proofs for γ_+ and γ_- are similar, we omit the index $\pm$ in the following. It is easy to see that γ is a G-map and satisfies $\beta(\gamma(\xi)) = \xi$ for all $\xi \in \mathbb{T}$. In order to see the measurability of γ, we define

$$F : \mathbb{T} \to \mathcal{K}(\Omega(人(W))) : \xi \mapsto \beta^{-1}(\xi),$$

where $\mathcal{K}(\Omega(人(W)))$ denotes the space of compact and non-empty subsets of $\Omega(人(W))$ equipped with the Hausdorff metric. Since β is continuous and $\Omega(人(W))$ is compact, the map F is upper semicontinuous and hence measurable. Now Lusin's Theorem yields the existence of compact set $K_n \subseteq \mathbb{T}$ with

$$\Theta_{\mathbb{T}}(K_n) > 1 - \frac{1}{n}$$

on which F is continuous. Hence, $\gamma|_{K_n}$ is also continuous. Thus, γ is measurable with respect to the completion of the σ-algebra of the Borel sets of $\mathbb{T}$. □

Now we consider the windows W and V constructed in Section 6.3. By construction of W, all critical fibres of β contain exactly two elements Γ_- and Γ_+ such that $\Gamma_- \subsetneq \Gamma_+$. Hence, by Proposition 6.27 above, the dynamical hull $(\Omega(人(W)), \mathbb{R})$ allows for two invariant graphs $\gamma_\pm$ mapping each $\xi \in \mathbb{T}$ to the maximal/minimal element of $\beta^{-1}(\xi)$.

In case of V, by Lemma 6.6, $(\Omega(人(V)), \mathbb{R})$ allows for one invariant graph γ mapping each $\xi \in \mathbb{T}$ to the maximal element of $\beta^{-1}(\xi)$.

Moreover, taking into account that $\beta : \Omega(人(W)) \to \mathbb{T}$ is almost everywhere 2-to-1, the proof of Lemma 6.8 yields that the associated graph measures $\mu_{\gamma_\pm}$ of $(\Omega(人(W)), \mathbb{R})$ are the only ergodic measures of $(\Omega(人(W)), \mathbb{R})$. Together with Proposition 6.26 this proofs the following statement.

Theorem 6.28. *Let $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ be a planar CPS and $W, V \subseteq \mathbb{R}$ constructed as above. Then the following holds.*

(i) $(\Omega(人(W)), \mathbb{R})$ *equipped with any ergodic measure has pure point spectrum with all eigenfunctions being continuous.*

(ii) $(\Omega(人(V)), \mathbb{R})$ *equipped with the unique invariant measure has pure point spectrum with all eigenfunctions being continuous.*

Concluding this chapter we want to point out a direct implication of this theorem concerning the studies of diffraction measures associated to Delone dynamical systems. Since we will not focus on this topic elsewhere in this thesis, we refer to [LS09], [LMS02], [Rob07] or [BLR07] for definitions and detailed discussions.

Corollary 6.29. *Let $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ be a planar CPS and $W, V \subseteq \mathbb{R}$ constructed as above. Then the following holds.*

(i) Let μ be an invariant measure on $(\Omega(人(W)), \mathbb{R})$. Then μ-almost every $\Gamma \in \Omega(人(W))$ has pure point diffraction spectrum.

(ii) Every $\Gamma \in \Omega(人(V))$ has pure point diffraction spectrum.

Proof. This is a direct consequence of [LMS02, Theorem 3.2, Theorem 4.1] and Theorem 6.28 above. □

Chapter 7

Tame implies Regular

In the last chapter of this part we are going to prove that tame implies regular. First, we discuss this for the case of model sets. Then we generalize the used methods to symbolic system and finally to arbitrary minimal group actions.

7.1 Tame implies Regular for Model Sets

Based on Theorem 3.15 we provide a criterion for non-tameness for Cut and Project Schemes analogous to Corollary 3.16. Let G be locally compact abelian second-countable group, $r > 0$ and $\Omega \subseteq \mathcal{U} = \mathcal{U}_r(G)$ an invariant subset with respect to the canonical G-action on $\mathcal{U}$ (compare Section 4.3). We say Ω *admits an (infinite) independence set* or *(infinite) free set* $S \subseteq G$ if there exists a uniformly discrete set $\Lambda \subseteq G$ such that

(IS1) $S \subseteq \Lambda$.

(IS2) For any subset $P \subseteq S$ there exists a $\Gamma \in \Omega$ such that

$$\Gamma \cap S = P.$$

Remark 7.1. In Section 5.1 we introduced the notion of embedded fullshifts. In fact, if $\mathcal{U}$ contains an embedded fullshift (Ξ, S) then Ξ admits an independence set S with $\Lambda = \bigcup_{\Gamma\in\Xi}\Gamma$. Then (IS1) follows by Lemma 5.1 and (IS2) is exactly property (FS3). However, since independence sets do not satisfy (FS1), the existence of independence sets does not yield the existence of embedded fullshifts.

Lemma 7.2. *Let G be a locally compact abelian second-countable group and $\Omega \subseteq \mathcal{U}$ a compact and translation-invariant subset. If Ω admits an infinite free set, then (Ω, G) is non-tame.*

Proof. Since Ω admits an infinite free set $S \subseteq G$ there exists some r-uniformly discrete set $\Lambda \subseteq G$ such that $S \subseteq \Lambda$. Put

$$U_0 = \{\Gamma \in \Omega \mid \Gamma \cap B_r(0) = \emptyset\}$$

and

$$U_1 = \{\Gamma \in \Omega \mid \Gamma \cap \mathrm{cl}(B_{r/2}(0)) \neq \emptyset\}.$$

By the assumptions, (U_0, U_1) form an independence pair. □

Hence, by Remark 7.1, we obtain the following statements for the dynamical systems constructed in Chapter 5.

Corollary 7.3. *Let $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ be a CPS.*

(i) Let $W = W(\omega) \subseteq \mathbb{R}$ be constructed as in Theorem 5.18. Then for $\mathbb{P}$-almost all $\omega \in \{0,1\}^{\mathbb{N}}$ and all $h \in \mathbb{R}$ the dynamical hull $(\Omega(\curlywedge(W(\omega) + h)), \mathbb{R}^N)$ is non-tame.

(ii) Let $W \subseteq \mathbb{R}$ be constructed as in Theorem 5.23. Then for all $h \in \mathbb{R}$ the dynamical hull $(\Omega(\curlywedge(W) + h), \mathbb{R}^N)$ is non-tame.

Now we provide a criterion for the existence of infinite independence sets which translates the dynamical problem into a purely geometric question about the structure of the window. Before we formulate this lemma, recall that

$$\mathcal{G}_W = \{h \in H \mid \sharp\beta^{-1}([g,h]_{\mathcal{L}}) = 1 \text{ for all } g \in G\}$$

was defined in Section 4.4.

Lemma 7.4. *Suppose $(G, H, \mathcal{L})$ is CPS with locally compact abelian and second-countable groups G and H. Let $W \subseteq H$ be a proper irregular window, i.e. $\Theta_H(\partial W) > 0$. Suppose there exists a relatively compact set $S^* \subseteq L^*$ such that for each $P^* \subseteq S^*$ we have*

$$H(S^*, P^*) = \bigcap_{s^* \in P^*} (W - s^*) \cap \bigcap_{s^* \in S^* \setminus P^*} (W^c - s^*) \cap (-\mathcal{G}_W) \neq \emptyset.$$

Then the set $S = \{s \in G \mid s^* \in S^*\}$ is free.

Proof. Let $P \subseteq S$ and choose $h \in H(S^*, P^*)$. Since $-h \in \mathcal{G}_W$ we obtain

$$\curlywedge(W - h) \in \Omega(\curlywedge(W)).$$

Further, for any $s \in S$, we have

$$s \in P \Leftrightarrow h \in W - s^* \Leftrightarrow s^* \in W - h \Leftrightarrow s \in \curlywedge(W - h).$$

Therefore, $\curlywedge(W - h) \cap S = P$. Note that if $P^* \neq \emptyset$, then $H(S^*, P^*) \subseteq W - P^* \subseteq W - S^*$. Hence, we may assume without loss of generality that the points h from above belong to the compact set

$$V = \mathrm{cl}(W - S^*) \cup \{h_0\} \subseteq H,$$

where h_0 is some point in $(W^c - S^*) \cap (-\mathcal{G}_W)$. Clearly,

$$\curlywedge(W - h) \subseteq \curlywedge(W - V) = \Lambda$$

for $h \in V$. Since $W - V$ is itself compact, the set Λ is uniformly discrete. As $P \subseteq S$ was arbitrary, we obtain that $S \subseteq \Lambda$ is a free set. □

For the next two statements we drop the restriction of H being abelian, since we will need this generalization for the discussions in Section 7.3. In the following, we will write the operation on H multiplicative. We denote the right Haar measure on H by Θ_H^r and the left Haar measure on H by Θ_H. It is well-known that the Haar measures of a locally compact second-countable group are outer regular. Let $C \subseteq H$ be a Borel set of positive measure. Set

$$\eta^C(\varepsilon) = \frac{\Theta_H^r(B_\varepsilon(C))}{\Theta_H^r(C)} - 1.$$

Then we have

$$\lim_{\varepsilon \to 0} \eta^C(\varepsilon) = 0.$$

Let $\Sigma_n = \{0,1\}^n$ and put $\Sigma_* = \bigcup_{n \in \mathbb{N}} \Sigma_n$. As before, by $|a|$ we denote the length of a word $a \in \Sigma_*$. In the following, we assume the metric $d = d_H$ to be left-invariant.

Lemma 7.5. *Let H be a locally compact and second-countable group. Suppose that $C \subseteq H$ is a Borel set with $\Theta_H^r(C) > 0$ and $(\xi_a)_{a \in \Sigma_*}$ is a collection of elements $\xi_a \in H$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that*

$$\varepsilon_n \geq \sup_{a \in \Sigma_n} d(0, \xi_a).$$

For $j \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ let

$$\delta_j^n = \sum_{l=j}^{n} \varepsilon_l.$$

Further, given $n \in \mathbb{N}$ and $a \in \Sigma_n$, let

$$\gamma_a = \prod_{j=1}^{n} \xi_{a_1,\ldots,a_j} = \xi_{a_1}\xi_{a_1,a_2}\cdots\xi_{a_1,\ldots,a_n}.$$

Then we have

$$\Theta_H^r\left(\bigcap_{a\in\Sigma_n} C\gamma_a^{-1}\right) \geq \Theta_H^r(C)\cdot\left(1-\sum_{j=1}^{n} 2^{j-1}\eta^C(\delta_j^n)\right)$$

for all $n \in \mathbb{N}$.

Proof. We proceed by induction on n.

Base case $n = 1$. In this case we have $\Sigma_1 = \{0,1\}$, $\gamma_0 = \xi_0$, $\gamma_1 = \xi_1$, $\delta_1^1 = \varepsilon_1$ and

$$d(0,\xi_0^{-1}) < \varepsilon_1 \text{ as well as } d(0,\xi_1^{-1}) < \varepsilon_1.$$

Both $C\gamma_0^{-1}$ and $C\gamma_1^{-1}$ are sets of measure $\Theta_H^r(C)$ which are contained in $B_{\delta_1^1}(C)$. Note that we have

$$\Theta_H^r(B_{\delta_1^1}(C)) = (1+\eta^C(\delta_1^1))\cdot\Theta_H^r(C).$$

This yields

$$\Theta_H^r((C\gamma_0^{-1})\cap(C\gamma_1^{-1})) \geq \Theta_H^r(C)\cdot(1-\eta^C(\delta_1^1)).$$

Inductive step $n \mapsto n+1$. Suppose the statement holds for some $n \in \mathbb{N}$ and all sets $C \subseteq H$ and all collections $(\xi_a)_{a\in\Sigma_*}$ as well as all sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ as above. Given $a \in \Sigma_*$, let $\xi_a' = \xi_{0a}$ and $\xi_a'' = \xi_{1a}$ and define γ_a', γ_a'' accordingly. Then

$$\bigcap_{a\in\Sigma_{n+1}} C\gamma_a^{-1} = \left(\underbrace{\left(\bigcap_{a'\in\Sigma_n} C\gamma_{a'}'^{-1}\right)}_{=I'}\xi_0^{-1}\right)\cap\left(\underbrace{\left(\bigcap_{a''\in\Sigma_n} C\gamma_{a''}''^{-1}\right)}_{=I''}\xi_1^{-1}\right).$$

By the induction hypothesis, both $I'\xi_0^{-1}$ and $I''\xi_1^{-1}$ are contained in $B_{\delta_1^{n+1}}(C)$ with measure greater equal

$$\Theta_H^r(C)\cdot\left(1-\sum_{j=1}^{n} 2^{j-1}\eta^C(\delta_{j+1}^{n+1})\right).$$

Hence, we obtain that

$$\begin{aligned}\Theta_H^r((I'\xi_0^{-1})\cap(I''\xi_1^{-1})) &\geq \Theta_H^r(C)\cdot\left(1-\eta^C(\delta_1^{n+1})-2\sum_{j=1}^{n} 2^{j-1}\eta^C(\delta_{j+1}^{n+1})\right)\\ &= \Theta_H^r(C)\left(1-\sum_{j=1}^{n+1} 2^{j-1}\eta^C(\delta_j^{n+1})\right).\end{aligned}$$

This completes the proof. □

Now we are going to prove the existence of an infinite set S^* that satisfies the assumptions of Lemma 7.4. In preparation for the arguments needed in Section 7.3 we state the following proposition in a more abstract form.

Proposition 7.6. *Suppose that H is a locally compact second-countable group with left Haar measure Θ_H. Moreover, suppose $V_0, V_1 \subseteq H$ are closed subsets such that* *(i)* V_0 *and* V_1 *are proper,*

(ii) $\mathrm{int}(V_0) \cap \mathrm{int}(V_1) = \emptyset$*,*

(iii) $\Theta_H(V_0 \cap V_1) > 0$.

Assume that $T \subseteq H$ *is a dense subgroup and* $\mathcal{G} \subseteq H$ *is a residual set. Then there exists an infinite set* $I \subseteq T$ *such that for all* $a \in \{0,1\}^I$ *there exists* $h \in \mathcal{G}$ *with the property that*

$$th \in \mathrm{int}(V_{a_t}) \text{ for } t \in I. \tag{7.1.1}$$

The same result holds if $H = \mathbb{S}$ *and (iii) is replaced by the assumption that* $V_0 \cap V_1$ *is a Cantor set.*

Proof. Let $\mathcal{G} = \bigcap_{n \in \mathbb{N}} G_n$, where each G_n is an open and dense subset of H. We will construct a sequence $(t_n)_{n \in \mathbb{N}}$ of points in T and a collection $(U_a)_{a \in \Sigma_*}$ of compact subsets of H with the following properties for all $n \in \mathbb{N}$ and $a \in \Sigma_n$:

(I1) $U_a \subseteq (t_n^{-1}\mathrm{int}(V_0)) \cap G_n$ if $a_n = 0$ and $U_a \subseteq (t_n^{-1}(\mathrm{int}(V_1))) \cap G_n$ if $a_n = 1$;

(I2) $U_{a0} \cup U_{a1} \subseteq U_a$.

This will prove the statement: if we define $I = \{t_n \mid n \in \mathbb{N}\}$, then for any given $a \in \{0,1\}^I$ we let $a^{(n)} = (a_{s1}, \ldots, a_{sn})$ and obtain from (I2) that $\bigcap_{n \in \mathbb{N}} U_{a^{(n)}}$ is a nested intersection of compact sets and therefore non-empty. By (I1), any $h \in \bigcap_{n \in \mathbb{N}} U_{a^{(n)}}$ has the property that $h \in \mathcal{G}$ and $t_n h \in \mathrm{int}(V_{a_{t_n}})$ for all $n \in \mathbb{N}$, as required by Equation 7.1.1.

We are going to construct $(t_n)_{n \in \mathbb{N}}$ and $(U_a)_{a \in \Sigma_*}$ by induction on $n = |a|$. Let us first specify some details. We will choose U_a as closed balls of the form $U_a = \mathrm{cl}(B_{r(|a|)}(\gamma_a))$ which we can ensure to be compact by choosing $r(.)$ sufficiently small. We set $\xi_0 = \gamma_0$, $\xi_1 = \gamma_1$ and $\xi_{a0} = \gamma_a^{-1}\gamma_{a0}$, $\xi_{a1} = \gamma_a^{-1}\gamma_{a1}$ for $a \in \Sigma_n$ and $n \geq 1$. By definition, we hence have

$$\gamma_a = \prod_{j=1}^{n} \xi_{a_1, \ldots, a_j},$$

which is consistent with the notation of Lemma 7.5. Further, we let $C = V_0 \cap V_1$. Observe that if $\Theta_H(C) > 0$, then $\Theta_H^r(C) > 0$, since Θ_H is absolutely continuous with respect to Θ_H^r. Now fix a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that

$$\sum_{j=1}^{\infty} 2^{j-1} \eta^C(\delta_j^\infty) < 1,$$

where the δ_j^n are defined as in Lemma 7.5. We moreover include the condition

(I3) $\sup_{a \in \Sigma_n} d(0, \xi_a) \leq \varepsilon_n$

in the inductive assumption. Note that this boils down to choosing γ_{a0} and γ_{a1} ε_{n+1}-close to γ_a in each step of the construction.

For the case $H = \mathbb{S}$ and $C = V_0 \cap V_1$ being a Cantor set, the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ and condition (I3) will not be needed. Instead, we will use the assumption

(I3') for all $a \in \Sigma_n$ we have $\partial U_a \subseteq t_{n+1}^{-1} C$

in this case.

Let us first consider the case that $\Theta_H(C) > 0$.

Base case $n = 1$. Choose $t_1 \in T$ and two open balls

$$U_0' = \mathrm{cl}(B_{r(1)}(\xi_0')) \subseteq \mathrm{int}(V_0) \cap t_1(G_1 \cap B_{\varepsilon_1}(0))$$

and

$$U_1' = \mathrm{cl}(B_{r(1)}(\xi_1')) \subseteq \mathrm{int}(V_1) \cap t_1(G_1 \cap B_{\varepsilon_1}(0)).$$

If we let $U_0 = t_1^{-1}U_0'$ and $U_1 = t_1^{-1}U_1'$, then (I1) and (I3) are satisfied for $n = 1$, and (I2) is still void.

Inductive step $n \mapsto n+1$. Suppose now that $t_1, \ldots, t_n$ and U_a for $a \in \bigcup_{j=1}^n \Sigma_j$ have been chosen and satisfy (I1)-(I3). Then Lemma 7.5 yields

$$\Theta_H^r\left(\bigcap_{a\in\Sigma_n} C\gamma_a^{-1}\right) \geq \Theta_H^r(C)\cdot\left(1-\sum_{j=1}^n 2^{j-1}\eta^C(\delta_j^\infty)\right) > 0.$$

In particular, the set on the left is non-empty and we can choose $h \in \bigcap_{a\in\Sigma_n} C\gamma_a^{-1}$. Clearly, $\gamma_a \in h^{-1}C$ for all $a \in \Sigma_n$. Now, we choose $t_{n+1} \in T$ close enough to h to guarantee that $t_{n+1}^{-1}C$ intersects $B_{r'(n+1)/2}(\gamma_a) \subseteq U_a$ for all $a \in \Sigma_n$, where $r'(n+1) = \min\{\varepsilon_{n+1}, r(n)\}$.

However, since points in C lie in the closure of the interior of both V_0 and V_1 this allows to find some $r(n+1) > 0$ as well as closed ball $U_{a0} = \mathrm{cl}(B_{r(n+1)}(\gamma_{a0}))$ and $U_{a1} = \mathrm{cl}(B_{r(n+1)}(\gamma_{a1}))$ with midpoints γ_{a0} and γ_{a1} ε_{n+1}-close to γ_a for all $a \in \Sigma_n$ such that (I1)-(I3) are satisfied for $n+1$.

Finally, if $H = \mathbb{S}$ and C is a Cantor set, we can proceed in a similar way without invoking Lemma 7.5. The crucial observation here is that if we choose some Δt_{n+1} sufficiently close to zero, then the rotation by $t_{n+1} = \Delta t_{n+1} t_n$ will send one of the endpoints of each U_a, $a \in \Sigma_n$, into $\mathrm{int}(U_a)$ (the left endpoints if Δt_{n+1} is locally to the right of zero and vice versa). Hence, we arrive at the same situation as in the first case. □

The observations above now lead to the following theorem.

Theorem 7.7. *Suppose that $(G, H, \mathcal{L})$ is a CPS with locally compact abelian and second-countable groups. If $W \subseteq H$ is a proper window with $\Theta_H(\partial W) > 0$ or if $H = \mathbb{S}$ and ∂W is a Cantor set, then $\Omega(⅄(W))$ admits an infinite free set $S \subseteq G$.*

Proof. First, consider the case $\Theta_H(\partial W) > 0$. Put $V_0 = \mathrm{cl}(W^c)$, $V_1 = W$, $T = L^*$ and $\mathcal{G} = -\mathcal{G}_W$. Note that we then have $\Theta_H(V_0 \cap V_1) = \Theta_H(\partial W) > 0$. Then Proposition 7.6 yields an infinite set $I \subseteq L^*$ such that for all $a \in \{0,1\}^I$ there exists some $h \in -\mathcal{G}_W$ such that

$$th \in \mathrm{int}(V_{a_t}), \text{ for } t \in I.$$

This means, if $a_t = 1$ for $t \in I$, there exists some $h \in -\mathcal{G}_W$ such that

$$h + I \subseteq V_1 = W.$$

By compactness of W this gives that I is relatively compact. Now, with $S^* = I$, the assumptions of Lemma 7.4 are fulfilled. This proves the theorem.

The proof works analogously if $H = \mathbb{S}$ and ∂W is a Cantor set. □

7.2 Tame implies Regular for Symbolic Systems

Consider the subshift $(\Sigma, \mathbb{Z})$ and denote by $\beta : \Sigma \to H$ the factor map onto its MEF. It is well-known that $(H, \mathbb{Z})$ is completely characterized by a transformation on H which we denote by ρ, that is, $n \cdot h = \rho^n(h)$ for $h \in H$ and $n \in \mathbb{Z}$. The basis for the direct application of the results from the last section to symbolic systems is provided by the following fact.

Proposition 7.8 ([BJL16]). *An almost automorphic subshift $(\Sigma, \mathbb{Z})$ is isomorphic to the system $(\Omega(⅄(W)), \mathbb{Z})$ obtained from the CPS $(\mathbb{Z}, H, \mathcal{L})$ with lattice $\mathcal{L} = \{(n, \rho^n(h_0)) \mid n \in \mathbb{Z}\}$, where*

(i) $h_0 \in H$ has a unique preimage under the factor map β;

(ii) $W = \beta([1])$, where $[1] = \{\xi \in \Sigma \mid \xi_0 = 1\}$.

Moreover, the window W is proper.

Together with Theorem 7.7 and Corollary 3.16 we directly obtain

Theorem 7.9. *If an almost automorphic subshift $(\Sigma, \mathbb{Z})$ is irregular, then it admits an infinite free set. In particular, it is non-tame. The same result holds if the MEF is an irrational circle rotation and $\beta([0]) \cap \beta([1])$ is a Cantor set.*

Remark 7.10. For the special case of Toeplitz flows this result has been established previously by Downarowicz ([Dow]).

7.3 Tame implies Regular for Minimal Group Actions

Now we generalize the discussions in the previous sections to the case of arbitrary minimal dynamical systems. Note that the following theorem provides an analogue to Theorem 7.7 for the case of general almost automorphic systems. However, this theorem does not imply Theorem 7.7 as a corollary, since the existence of an infinite free set does not follow directly from non-tameness.

Theorem 7.11. *Let X be a compact topological space and $G = (G, \cdot)$ a topological group. Suppose that (X, G) is almost automorphic. If (X, G) is tame, then it is a regular extension of its MEF.*

Proof. Let us denote the maximal equicontinuous factor of (X, G) by (H, G). As (H, G) is minimal and equicontinuous, Theorem 3.12 implies that (H, G) is a factor of $(\mathcal{E}(H), G)$ with factor map

$$\pi : \mathcal{E}(H) \to H.$$

Further, we denote the unique G-invariant measure on H by μ. Recall that $\mu = \Theta_{\mathcal{E}(H)} \circ \pi^{-1}$ and π is open (compare Remark 3.13).

Assume for a contradiction that β is not almost surely one-to-one with respect to the measure μ on H so that (X, G) is an irregular extension of (H, G). We aim to show the existence of an independence pair (U_0, U_1) for (X, G) which implies non-tameness by Theorem 3.15.

To that end, consider the space $\mathcal{K}(X)$ of compact subsets of X equipped with the Hausdorff metric $d_{\mathcal{H}}$. Consider the mapping

$$F : H \to \mathcal{K}(X) : \xi \mapsto \beta^{-1}(\xi).$$

Comparing the proof of Proposition 6.27, we infer that F is measurable. By Lusin's Theorem, we may therefore choose a compact set $K \subseteq H$ of positive measure such that $F|_K$ is continuous. Let $K_0 \subseteq K$ denote the topological support of the measure $\mu|_K$ (i.e., the essential closure of K). Then $\mu(K_0) = \mu(K) > 0$. Hence, by irregularity, we can find some $h_0 \in K_0$ such that $\sharp\beta^{-1}(h_0) > 1$. Moreover, we have that $\mu(V \cap K) > 0$ for every neighbourhood V of h_0.

Choose $\xi_0, \xi_1 \in \beta^{-1}(h_0)$ such that $\xi_0 \neq \xi_1$. Further, put $\varepsilon = \frac{d(\xi_0,\xi_1)}{4}$ and $U_i = \mathrm{cl}(B_\varepsilon(\xi_i))$ for $i \in \{0, 1\}$. We aim to show that (U_0, U_1) is an independence pair for (X, G), i.e. that there is an infinite set $I \subseteq G$ such that for any $a \in \{0, 1\}^I$ there exists some $\xi \in X$ with

$$t\xi \in U_{a_t} \text{ for } t \in I. \tag{7.3.1}$$

Let $V_0 = \beta(U_0)$ and $V_1 = \beta(U_1)$. Since β is almost one-to-one, both sets are proper. Since points with singleton fibres are dense, V_0 and V_1 have disjoint interior.

Due to continuity of F on K, we can choose $\delta > 0$ such that for any $h \in B_\delta(h_0) \cap K$ we have

$$d_{\mathcal{H}}(F(h), F(h_0)) < \varepsilon.$$

This yields that the fibre $F(h) = \beta^{-1}(h)$ intersects both U_0 and U_1, so that $h \in V_0 \cap V_1$. Therefore, $B_\delta(h_0) \cap K \subseteq V_0 \cap V_1$, so that

$$\mu(V_0 \cap V_1) \geq \mu(B_\delta(h_0) \cap K) > 0.$$

Set $V_0^{\mathcal{E}} = \pi^{-1}(V_0)$ and $V_1^{\mathcal{E}} = \pi^{-1}(V_1)$. Since π is open, both sets are proper. Thus, the assertions of Proposition 7.6 are met by $V_i^{\mathcal{E}} \subseteq \mathcal{E}(H)$ with $\mathcal{G} = \pi^{-1}(\beta(X_0))$, where X_0 denotes the set of injectivity points of β. Since π is open, we obtain that $\mathcal{G}$ is residual.

Hence, we obtain a set $I \subseteq G$ and for each $a \in \{0, 1\}^I$ a point $h' \in \mathcal{G}$ such that $th' \in \mathrm{int}(V_{a_t}^{\mathcal{E}})$ for $t \in I$ and hence

$$th \in \mathrm{int}(V_{a_t}) \text{ for } t \in I \tag{7.3.2}$$

for $h = \pi(h') \in \beta(X_0)$. However, since h has a unique preimage under β, Equation 7.3.2 directly implies Equation 7.3.1 so that (U_0, U_1) is an independence pair as claimed. $\square$

Part III

Dynamical Cut and Project Schemes

Chapter 8

Introduction to Dynamical Cut and Project Schemes

In this final part, we aim to generalize the Cut and Project method. This first chapter is dedicated to provide the necessary background for understanding the concept of dynamical Cut and Project Schemes.

8.1 Cocycles

Let $T = (T, \cdot)$ and $G = (G, +)$ be abelian groups. Let X be a compact topological space and assume (X, T) is a topological dynamical system. A mapping

$$\varphi : T \times X \to G : (t, x) \mapsto \varphi_t(x)$$

satisfying

$$\varphi_{st}(x) = \varphi_s(x) + \varphi_t(s \cdot x) \tag{CoEq}$$

for all $s, t \in T$ and $x \in X$ is called *cocycle*. We refer to (CoEq) as *cocycle equation*.

As a first example, consider the case $T = \mathbb{Z}$ and $G = \mathbb{R}$. According to previous discussions in Section 3.1 we may express the $\mathbb{Z}$-action via a transformation

$$H : X \to X.$$

Then every mapping $f : X \to \mathbb{R}$ induces a cocycle $\varphi : \mathbb{Z} \times X \to G$ via

$$\varphi_n(x) = \begin{cases} \sum_{i=0}^{n-1} f(H^i(x)) \text{ for } n > 0 \\ \\ -\sum_{i=n}^{-1} f(H^i(x)) \text{ otherwise.} \end{cases} \tag{8.1.1}$$

In particular we have that $\varphi_0(x) = 0$. If no confusion arises, we will use the convention $\sum_{i=0}^{n-1} f(.) = -\sum_{i=n}^{-1} f(.)$ in case of $n \leq 0$.

We may generalize the above to higher dimensional acting groups (compare also [Bjö09] and [Frą00]). Put $T = \mathbb{Z}^N$ and suppose the corresponding action is given via a mapping $H : X \to X$ satisfying

$$H^{(n_1,\ldots,n_N)} = H_1^{n_1} \circ \ldots \circ H_N^{n_N}$$

and the $H_1, \ldots, H_N$ are commuting (compare the discussions in Section 6.4 for an example of such a $\mathbb{Z}^N$-action given by N rationally independent rotations on the circle). Suppose we have given N measurable functions

$$f_i : X \to G,\ i = 1, \ldots, N$$

which satisfy

$$f_i(H_j(x)) + f_j(x) = f_i(x) + f_j(H_i(x))$$

for all $x \in X$ and $i, j \in \{1, \dots, N\}$. Put $\varphi_{e_i} = f_i$, where e_i denotes the i-th canonical basis vector in $\mathbb{Z}^N$. We define recursively a sequence of functions via

$$\varphi_{n+m}(x) = \varphi_n(x) + \varphi_m(H^n(x)),$$

where $n, m \in \mathbb{Z}^N$ and $x \in X$. Clearly, φ satisfies (CoEq) and is thus a cocycle.

Moreover, a straightforward calculation shows that φ may be represented as

$$\varphi_{(n_1,\dots,n_N)}(x) = \sum_{i=1}^{N} \left(\sum_{j=0}^{n_i-1} f_i \left(H_i^j \circ H_{i+1}^{n_{i+1}} \circ \dots \circ H_N^{n_N}(x) \right) \right) \tag{8.1.2}$$

(recall that we use the convention $\sum_{j=0}^{n_i-1} f(.) = -\sum_{j=n_i}^{-1} f(.)$ in case of $n_i \leq 0$).

We refer to φ as *induced cocycle* and call the functions $f_1, \dots, f_N$ *generators*. Observe that each cocycle $\varphi : \mathbb{Z}^N \times X \to G$ is induced by the functions $f_i = \varphi_{e_i} : X \to G$, $i = 1, \dots, N$.

8.2 Delone Cocycles and Dynamical Cut and Project Schemes

Let $G = (G, +)$ be a locally compact abelian and second-countable group and let $T = (T, \cdot)$ be a discrete subgroup of some metrizable group T'. Throughout the next chapters we will assume that T carries a proper metric (recall Lemma 2.2 and Remark 2.3). Let X be a compact topological space and assume that T is acting minimally on X via

$$T \times X \to X : (t, x) \mapsto t \cdot x$$

and let $\varphi : T \times X \to G$ be a cocycle. We call the tuple $\mathcal{D} = (X, T, G, \varphi)$ a *dynamical Cut and Project Scheme (DCPS)*. Given a *window* $W \subseteq X$ and a *starting point* $x_0 \in X$, a DCPS gives rise to a *(dynamical) model set* in G which is given by

$$\curlywedge_{x_0}^{T}(W) = \{\varphi_t(x_0) \mid t \cdot x_0 \in W\}.$$

As in the case of classical CPS we will also refer to translates $\curlywedge_{x_0}^{T}(W) - g$, $g \in G$, as *model sets*. Further, sometimes we refer to the group G as *physical space*. In case of $G = \mathbb{R}^N$ we refer to the DCPS as *Euclidean*.

The cocycle might be chosen arbitrarily (for instance, $\varphi_s(x) = 0$ for all $s \in T$, $x \in X$) which might lead to undesired properties of $\curlywedge_{x_0}^{T}(W)$ (which would equal $\{0\}$ in this case). Thus, most of the remaining section is dedicated to discuss several additional properties of cocycles and their effects on the corresponding dynamical model sets. Concluding this discussion we will provide examples of cocycles satisfying those properties.

We call $\varphi : T \times X \to G$ a *Delone cocycle* if it satisfies the following conditions:

(D1) there exists a constant $c > 0$ such that for all $x \in X$ and $s, t \in T$ with $s \neq t$ holds

$$d_G(\varphi_s(x), \varphi_t(x)) \geq c.$$

(D2) for every $x \in X$ there exists some $C > 0$ such that

$$\varphi_T(x) \cap B_C(g) \neq \emptyset$$

for all $g \in G$.

(D3) for all $x \in X$ and for all $R > 0$ there exists some $\rho > 0$ such that

$$d_T(s, t) < R \Longrightarrow d_G(\varphi_s(x), \varphi_t(x)) < \rho$$

for all $s, t \in T$.

The following observation points out why φ is referred to as Delone cocycle.

Proposition 8.1. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Assume that φ is a Delone cocycle. Then*

(i) $\curlywedge_{x_0}^{T}(W)$ *is uniformly discrete.*

(ii) $\curlywedge_{x_0}^{T}(W)$ *is relatively dense.*

In other words, $\curlywedge_{x_0}^{T}(W)$ *is a Delone set.*

Proof. (i) follows immediately from the cocycle property (D1). To show (ii) let $g \in G$. By (D2), there exists some $t \in T$ such that $\varphi_t(x_0) \in B_C(g)$. Since T is acting minimally on X and $\mathrm{int}(W) \neq \emptyset$, by Lemma 3.2 x_0 is an almost periodic point. This means that the set of return times

$$N(x_0, W) = \{u \in T \mid u \cdot x_0 \in W\}$$

is syndetic. Then we may find some $R > 0$ such that $T = N(x_0, W) \cdot \mathrm{cl}(B_R(0))$ (note that $\mathrm{cl}(B_R(0)) \subseteq T$). Hence, there exists some $s \in N(x_0, W)$ such that $d_T(s, t) < R$. Then, by (D3), there exists $\rho > 0$ such that $d_G(\varphi_s(x_0), \varphi_t(x_0)) < \rho$. Thus, we obtain $d_G(\varphi_s(x_0), g) < C + \rho$. Since $\varphi_{N(x_0,W)}(x_0) = \curlywedge_{x_0}^{T}(W)$, the set $\curlywedge_{x_0}^{T}(W)$ is relatively dense in G with constant $C + \rho$. $\square$

Remark 8.2. As seen in the previous proof, uniform discreteness of dynamical model sets follows directly from property (D1). In contrast, relative denseness additionally depends on W, but holds whenever $\mathrm{int}(W) \neq \emptyset$.

Next we want to point out a crucial connection between translations of model sets in G and iterations of the associated starting points in X.

Lemma 8.3. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with window $W \subseteq X$ and starting point $x_0 \in X$. Then*

$$\curlywedge_{t\cdot x_0}^{T}(W) = \curlywedge_{x_0}^{T}(W) - \varphi_t(x_0)$$

for all $t \in T$. In particular we obtain $P(R, \varphi_t(x_0)) = \curlywedge_{t\cdot x_0}^{T}(W) \cap B_R(0)$.

Proof. Let $t \in T$. The cocycle equation (CoEq) yields

$$\begin{aligned}\curlywedge_{t\cdot x_0}^{T}(W) &= \{\varphi_s(t \cdot x_0) \mid s(t \cdot x_0) \in W\} \\ &= \{\varphi_{ts}(x_0) \mid st \cdot x_0 \in W\} - \varphi_t(x_0) \\ &= \curlywedge_{x_0}^{T}(W) - \varphi_t(x_0).\end{aligned}$$

Hence, the first assumption holds. The second identity follows as an immediate consequence. $\square$

In the following we are going to introduce a few more properties for Delone cocycles. We say a Delone cocycle φ satisfies the *distance property* if

(Dist) for all $R > 0$ there exists some $\rho > 0$ such that for all $x \in X$ and $t \in T$ holds

$$d_T(0, t) > \rho \Longrightarrow d_G(\varphi_t(x), 0) > R.$$

Remark 8.4. In literature, this property is also known as *uniform growth of φ* or φ being a *covering cocycle* (compare [KMMS98], [FKMS93]).

Further, the Delone cocycle is said to be *piecewise constant* if

(PwCo) for all $s \in T$ the map $\varphi_s(x)$ is piecewise constant for all $x \in X$, that is, for each $s \in T$ there exists a partition $\mathcal{P}(s)$ of X (compare Remark 3.29) such that

(PwCo1) $\sharp\mathcal{P}(s) < \infty$, (PwCo2) $\sharp\varphi_s(P) = 1$ and $\varphi_s(P) \neq 0$ for all $P \in \mathcal{P}(s)$ with $s \neq 0$.

Remark 8.5. (i) An immediate consequence of the cocycle equation is $\varphi_0(x) = 0$ for all $x \in X$. This is the reason we exclude the case $s = 0$ in (PwCo2).

(ii) Due to (PwCo2) and $\varphi_s(x) = \varphi_t(x) + \varphi_{s-t}(t \cdot x)$ we infer that $\varphi_s(x) = \varphi_t(x)$ if and only if $s = t$.

A Delone cocycle which satisfies both (Dist) and (PwCo) is called *FLC cocycle*.

Proposition 8.6. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with window $W \subseteq X$ and starting point $x_0 \in X$. If φ is an FLC cocycle, then $人^T_{x_0}(W)$ has FLC.*

Proof. Fix $R > 0$ and put $B_R = \mathrm{cl}(B_R(0))$. Let $t \in T$. By Lemma 8.3 we have $P(R, \varphi_t(x_0)) = 人^T_{t\cdot x_0}(W) \cap B_R$. By uniform discreteness of $人^T_{x_0}(W)$ there exists some $K \in \mathbb{N}$ such that

$$P(R, \varphi_t(x_0)) = \{\varphi_{s_1}(t \cdot x_0), \dots, \varphi_{s_K}(t \cdot x_0)\}.$$

Thus, to each patch we may associate a set of corresponding times which we will denote by $I(P(R, \varphi_t(x_0))) = \{s_1, \dots, s_K\}$. Due to (Dist) there exists $\rho = \rho(R) > 0$ such that

$$d_G(0, \varphi_u(x)) \leq R \Longrightarrow d_T(0, u) \leq \rho$$

for all $x \in X$ and $u \in T$. Since T is equipped with a proper metric, the set $D(R) = \{u \in T \mid d_T(0, u) \leq \rho\}$ is finite and we have

$$I(P) \subseteq D(R) \text{ for all } R\text{-patches } P \in \mathcal{P}(人^T_{x_0}(W)). \tag{8.2.1}$$

In particular each R-patch consists of a maximum of $\sharp D(R)$ points.

Now fix some R-patch $P = 人^T_{t\cdot x_0}(W) \cap B_R$ and let $p \in P$. Then $p = \varphi_s(t \cdot x_0)$ for some $s \in I(P)$. On the other hand, due to φ_s satisfying (PwCo1) and (PwCo2), there exists some $N(s) \in \mathbb{N}$ and values $c_1(s), \dots, c_{N(s)}(s) \in G$ such that $p \in \{c_1(s), \dots, c_{N(s)}(s)\}$. Together with finiteness of $D(R)$ and (8.2.1) this implies that there occur just finitely many different configurations of points of $人^T_{x_0}(W)$ in a ball of radius R. Hence, $\sharp\{人^T_{t\cdot x_0}(W) \cap B_R \mid t \cdot x_0 \in W\} < \infty$. □

In some cases it might become handy for an FLC cocycle φ to satisfy a few more properties. We say a cocycle φ is an *aperiodic FLC cocycle* if

(APwCo) it is an FLC cocycle such that

(APwCo1) for all $s \in T \setminus \{0\}$ we have $\sharp\mathcal{P}(s) \geq 2$ and there exist at least two distinct set $P, Q \in \mathcal{P}(s)$ satisfying $\varphi_s(P) \neq \varphi_s(Q)$ as well as $\varphi_s(P) \neq -\varphi_s(Q)$,

(APwCo2) for all $s, t \in T$ and $x, y \in X$ we have

$$\varphi_s(x) = \varphi_t(y) \Longrightarrow s = t.$$

Remark 8.7. (i) Recall that our definition of a partition immediately yields that $\mathrm{int}(P) \neq \emptyset$ for all partition elements P (compare Remark 3.29).

(ii) In [FKMS93], cocycles satisfying (Dist) as well as (APwCo2) are referred to as *embedding cocycles*.

It turns out that aperiodic FLC cocycles might be used to describe aperiodic point sets.

Lemma 8.8. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. If φ is an aperiodic FLC cocycle with respect to W, then $人^T_{x_0}(W)$ is aperiodic.*

Proof. Let $R > 0$. Put $B_R = \mathrm{cl}(B_R(0))$ and $\Lambda = 人^T_{x_0}(W)$. Let $g \in G$ such that $\Lambda - g = \Lambda$. In particular we have

$$P(R, g) = P(R, 0).$$

We may write $P(R,g) = \{\varphi_{t_1}(x_0)-g, \ldots, \varphi_{t_N}(x_0)-g\}$ as well as $P(R,0) = \{\varphi_{s_1}(x_0), \ldots, \varphi_{s_N}(x_0)\}$, where $N = N(R) \in \mathbb{N}$. Further, we label $t_i, s_i \in T$ such that we have $\varphi_{t_i}(x_0) - g = \varphi_{s_i}(x_0)$. Then the cocycle equation yields

$$g = \varphi_{t_i - s_i}(s_i \cdot x_0) \tag{8.2.2}$$

for all $i = 1, \ldots, N$. By (APwCo2) there exists some $c \in T$ such that $t_i - s_i = c$ for all $i = 1, \ldots, N$. By (APwCo1), we may find sets $P, Q \in \mathcal{P}(c)$ such that

$$\varphi_c(P) \neq \pm\varphi_c(Q). \tag{8.2.3}$$

Since (X,T) was supposed to be minimal, without loss of generality (otherwise we have to increase R) we may find distinct indices $i_1, i_2 \in \{1, \ldots, N\}$ such that we have $s_{i_1} \cdot x_0 \in \mathrm{int}(P)$ and $s_{i_2} \cdot x_0 \in \mathrm{int}(Q)$. Together with (8.2.3) this contradicts (8.2.2) except in case $c = 0$. Hence, $g = 0$. ☐

More generally, we say a Delone cocycle φ is *piecewise continuous* if

(PwCts) for all $s \in T$ there exists a partition $\mathcal{P}(s)$ of X such that

(PwCts1) $\sharp\mathcal{P}(s) < \infty$,

(PwCts2) $\sharp\varphi_s(P) = \infty$ and $\varphi_s|_P$ is continuous for all $P \in \mathcal{P}(s)$.

Assuming a Delone cocycle φ satisfies (Dist) and (PwCts), the corresponding model set $人^T_{x_0}(W)$ has not to satisfy (FLC) anymore. For this reason, we will refer to such cocycles as *non-FLC cocycles*.

Suppose T is even a lattice in G. We say the cocycle φ satisfies the *uniform distortion property* if

(UDP) there exists an injective homomorphism $H : T \to G$ such that for all $x \in X$ and $s \in T$ holds

$$\varphi_s(x) = H(s) + o_x(s).$$

Here, $o_x : T \to G$ denotes a mapping with parameter $x \in X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that that for all $s \notin B_\delta(0)$ holds

$$d_G(0, o_x(s)) \leq \varepsilon d_G(0, s). \tag{8.2.4}$$

In case there exists a $C > 0$ such that $o_x(s) \in \mathrm{cl}(B_C(0))$ for all $x \in X$ and $s \in T$, we say φ satisfies the *strict uniform distortion property* (or (SUDP) for short).

For reasons becoming clear in Section 9.2, we will refer to a cocycle satisfying (Dist), (PwCo) and (UDP) (or (SUDP)) as *(strict) UPF cocycle* (or *(strict) uniform patch frequency-cocycle*).

Concluding this chapter, we want to provide a simple class of examples for FLC cocycles. To that end, consider a dynamical system $(X, \mathbb{Z})$, where the $\mathbb{Z}$-action is given via an homeomorphism $H : X \to X$. Let $\mathcal{P} = (P_i)_{j=1}^K$ be a finite partition of X. Now consider the step function

$$f : X \to \mathbb{R} : x \mapsto \sum_{j=1}^{K} a_j \chi_{P_j}(x), \tag{8.2.5}$$

where the coefficients $a_j \in \mathbb{R}$ are chosen such that

- $a_j \neq 0$,
- $\mathrm{sgn}(a_j) = \mathrm{sgn}(a_{j'})$ for all $j, j' \in \{1, \ldots, K\}$.

Following the discussions in Section 8.1, the map f is the generator of a cocycle $\varphi : \mathbb{Z} \times X \to \mathbb{R}$ given as in (8.1.1). In the following, fix an arbitrary point $x_0 \in X$.

First, we want to point out the following identity. Let $n, m \in \mathbb{Z}$. By the cocycle equation, we obtain

$$\varphi_n(x_0) - \varphi_m(x_0) = \varphi_{n-m}(H^m(x_0)) = \sum_{j=1}^{K} l_j a_j,$$

where $l_j = \sum_{i=0}^{n-m-1} \chi_{P_j}(H^i(H^m(x_0)) \in \mathbb{Z}$. Observe that these coefficients satisfy

- $\sum_{j=1}^{K} l_j = n - m$,
- $\operatorname{sgn}(l_1) = \ldots = \operatorname{sgn}(l_K) = \operatorname{sgn}(n-m)$.

Now observe that we have $\varphi_{n+1}(x_0) = \varphi_n(x_0) + \varphi_1(H^n(x_0))$ for any $n \in \mathbb{Z}$. This means, that in each iteration φ grows at least by the value $r = \min_{j=1}^{K} |a_j|$. However, since all a_j have the same sign, we immediately obtain that between any two values $\varphi_n(x_0)$ and $\varphi_m(x_0)$ is at least a distance r, i.e.,

$$|\varphi_n(x_0) - \varphi_m(x_0)| = \left| \sum_{j=1}^{K} k_j a_j \right| \geq r$$

for all distinct $n, m \in \mathbb{Z}$. Thus, φ satisfies (D1).

Regarding (D2), we notice that $\varphi_{\pm n}(x_0) \to \pm\infty$ as $n \to \infty$. By putting $R = \max_{j=1}^{K} |a_j|$, we obtain that

$$\varphi_{\mathbb{Z}}(x_0) \cap B_{2R+\varepsilon}(p) \neq \emptyset$$

for some $\varepsilon > 0$ and all $p \in \mathbb{R}$.

Let $C > 0$ such that $|n - m| < C$ for $n, m \in \mathbb{Z}$. Without loss of generality we may assume $n \geq m$. Again, put $R = \max_{j=1}^{K} |a_j|$. Then we may calculate

$$\begin{aligned} |\varphi_n(x_0) - \varphi_m(x_0)| &\leq \sum_{i=0}^{n-m-1} \left(\sum_{j=1}^{K} |a_j| \cdot \left| \chi_{P_j}(H^i(x_0)) \right| \right) \\ &\leq \sum_{i=0}^{n-m-1} K \cdot R \leq |n-m| \cdot K \cdot R < K \cdot R \cdot C. \end{aligned}$$

Thus, φ satisfies (D3) and is hence a Delone cocycle.

Now we want to show that φ satisfies (Dist). To that end, let $R > 0$ be arbitrary and define

$$\rho = \frac{R}{\min_{j=1}^{K} |a_j|}.$$

It is not hard to see that we have

$$|n| \cdot \min_{j=1}^{K} |a_j| \leq |\varphi_n(x)|$$

for arbitrary $x \in X$ and $n \in \mathbb{Z}$. Hence, if $|\varphi_n(x)| \leq R$, then $|n| \leq \rho$. Thus, φ satisfies (Dist).

To show that φ is an FLC cocycle, we aim to proof that (PwCo) holds for φ. To that end, consider first the case $n = 0$. Put $\mathcal{P}(0) = \{X\}$. Clearly, $\mathcal{P}(0)$ satisfies (PwCo1) as well as (PwCo2). Since $\varphi_0(x) = 0$ for all $x \in X$, (PwCo2) holds.

In case of $n = 1$, we may put $\mathcal{P}(1) = \mathcal{P}$. By our assumptions on $\mathcal{P}$, (PwCo1) and (PwCo2) holds. Now consider the case $n = 2$. Recall that

$$\varphi_2(x) = \varphi_1(x) + \varphi_1(H(x)).$$

Observe that

$$\varphi_1(H(x)) = \sum_{j=1}^{K} a_j \chi_{P_j}(H(x)) = \sum_{j=1}^{K} a_j \chi_{H^{-1}(P_j)}(x),$$

which yields

$$\varphi_2(x) = \sum_{j=1}^{K} a_j \cdot \chi_{P_j}(x) + a_j \cdot \chi_{H^{-1}(P_j)}(x).$$

Since H is a homeomorphism, the collection $H^{-1}(\mathcal{P}) = \{H^{-1}(P_1), \ldots, H^{-1}(P_K)\}$ is also a finite partition of X. Thus, it is not hard to see that $\varphi_2(x) = a_i + a_j$ if and only if $x \in P_i \cap H^{-1}(P_j)$. Moreover, $\mathcal{P}(2) = \{P_i \cap H^{-1}(P_j) \mid i, j \in \{1, \ldots, K\}\}$ is a partition of X satisfying (PwCo1) as well as (PwCo2). By proceeding successively and using

$$\varphi_n(x) = \sum_{i=0}^{n-1} \varphi_1(H^i(x)),$$

for each $n \in \mathbb{N}$ we obtain a finite partition of sets

$$\mathcal{P}(n) = \left\{ \bigcap_{i=0}^{n-1} H^{-i}(P_{j(i)}) : j(i) \in \{1, \ldots, K\} \right\}$$

of X such that $\sharp\varphi_n(P) = 1$ for all $P \in \mathcal{P}(n)$. Since the same construction works analogously for negative $n \in \mathbb{Z}$, we obtain that φ indeed satisfies (PwCo). Hence, φ is an FLC cocycle.

To give an example for an aperiodic FLC cocycle, we will additionally assume that all a_j are rationally independent. In particular we have $a_i \neq a_j$ for all $i \neq j$ and thus (APwCo1) holds.

Let $n, m \in \mathbb{Z}$ and $x, y \in X$. Since all coefficients have the same sign, in the following it is sufficient to focus on $n, m \in \mathbb{N}$. Then $\varphi_n(x) = \varphi_m(y)$ yields

$$\sum_{i=0}^{n-1} \sum_{j=1}^{K} a_j \chi_{P_j}(H^i(x)) - \sum_{j=0}^{m-1} \sum_{j=1}^{K} a_j \chi_{P_j}(H^i(y)) = 0$$

which is equivalent to

$$\sum_{j=1}^{K} a_j \left(\sum_{i=0}^{n-1} \chi_{P_j}(H^i(x)) - \sum_{j=0}^{m-1} \chi_{P_j}(H^i(y)) \right) = 0.$$

Put $k_j = \sum_{i=0}^{n-1} \chi_{P_j}(H^i(x))$ and $l_j = \sum_{i=0}^{m-1} \chi_{P_j}(H^i(y))$, respectively. By rational independence, we obtain $k_j = l_j$ for each $j \in \{1, \ldots, K\}$. Since $\sum_{j=1}^{K} k_j = n$ and $\sum_{j=1}^{K} l_j = m$, we obtain $n = m$. This shows (APwCo2).

Finally, we aim to give an example for FLC cocycles satisfying UDP. To that end, we drop the additional assumptions we made previously in the case of (APwCo). Instead, we assume that $(X, \mathbb{Z})$ carries a unique ergodic measure μ. Additionally, we claim that $\mu(\partial P_j) = 0$ for all $P_j \in \mathcal{P}$.

Then

$$L : \mathbb{Z} \to \mathbb{R} : n \mapsto n \cdot \int_X f \, \mathrm{d}\mu$$

is a well-defined, linear and injective function. Now consider the map

$$R_x(n) = \varphi_n(x) - n \cdot \int_X f \, \mathrm{d}\mu.$$

Clearly, $\varphi_n(x) = L(n) + R_x(n)$. By Lemma 3.28 we obtain

$$\begin{aligned}\left|\frac{1}{n}R_x(n)\right| &= \left|\frac{1}{n}\left(\varphi_n(x) - n\cdot\int_X f\,\mathrm{d}\mu\right)\right| \\ &= \left|\frac{1}{n}\sum_{i=1}^{n-1} f(H^i(x)) - \int_X f\,\mathrm{d}\mu\right| \to 0 \text{ as } n\to\infty\end{aligned}$$

for every $x \in X$. Hence, $R_x(n) = o_x(|n|)$ in the sense of (8.2.4). Thus, φ is a UPF cocycle.

In Section 9.3 we will discuss a condition for the dynamical system $(X, \mathbb{Z})$ (namely, the partition corresponding with f consists of bounded remainder sets) such that our cocycle φ satisfies SUDP (compare also Remark 9.15).

Altogether, we have proven

Lemma 8.9. *Let $(X, \mathbb{Z})$ be a dynamical system and let φ denote the cocycle induced by a step function as given in (8.2.5). Then the following holds.*

(i) The cocycle φ is an FLC cocycle.

(ii) If $(X, \mathbb{Z}, \mu)$ is uniquely ergodic, the cocycle φ is an UPF cocycle. If the associated partition of f consists of bounded remainder sets, then φ is a SUPF cocycle.

(iii) If the coefficients of the induced cocycle are rationally independent and $W \subseteq X$ is chosen such that its intersections with at least two partition elements is non-empty, then φ is an aperiodic FLC cocycle with respect to W.

Remark 8.10. (i) As seen in the previous construction, minimality of $(X, \mathbb{Z})$ (or, more general, minimality of (X, T)) is not required for a cocycle to satisfy the Delone property. In the previous example, in case of $a_j \neq a_{j'}$ for all $j \neq j'$, we might obtain an infinite dynamical model set $\curlywedge_{x_0}^T(W)$, as long as $W \cap P \neq \emptyset$ for at least two different $P \in \mathcal{P}$. However, in such a case $\curlywedge_{x_0}^T(W)$ might not be aperiodic anymore.

(ii) Based on the previous example, we might construct FLC cocycles $\mathbb{Z}^N \times X \to \mathbb{R}^N$ generated by step functions which assign vectors to partition elements instead of scalars (compare Section 11.2). However, showing that such a cocycle is indeed an FLC cocycle is in general more difficult then the previous example.

(iii) As long as every point $x \in X$ is generic, the method used above provides a decomposition of φ into two functions such that (UDP) is satisfied.

Chapter 9

Geometric Properties

In the following, we will discuss several geometric aspects of dynamical Cut and Project Schemes. First, we aim to discuss the influence of the chosen cocycle and window on geometric properties of the associated dynamical model set. Afterwards, we discuss the interplay between properties of the underlying dynamical system and the corresponding model sets in the setting of Euclidean dynamical Cut and Project Schemes.

Recall that, by our assumptions on the groups G and T, these groups admit a proper metric, i.e., all balls with respect to this metric are relatively compact. Throughout this chapter we will always assume that our groups are equipped with this metric.

9.1 Repetitivity

In this section we first want to reformulate Lemma 4.12(i) in terms of dynamical CPS. To that end, we will discuss DCPS with FLC cocycles. Following this, we will consider DCPS with non-FLC cocycles and state a lemma about almost repetitivity of non-FLC sets.

Before we discuss repetitivity let us introduce some notation. Consider a DCPS $\mathcal{D} = (X, T, G, \varphi)$ with window $W \subseteq X$ and an arbitrary point $x \in X$. We say x *is generic with respect to* W if $\mathcal{O}_T(x) \cap \partial W = \emptyset$. If φ satisfies (PwCo) we say x *is generic with respect to* φ if for all $s \in T$ holds $x \notin \bigcup_{P \in \mathcal{P}(s)} \partial P$.

Lemma 9.1. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with proper window $W \subseteq X$. Then the following holds:*

(i) There exists $x \in X$ such that x is generic with respect to W.

(ii) If φ satisfies (PwCo) there exists $x \in X$ such that x is both generic with respect to W and generic with respect to φ.

Proof. (i). Since T is countable and $\operatorname{int}(\partial W) = \emptyset$,

$$T \cdot \partial W = \bigcup_{t \in T} t \cdot \partial W$$

cannot agree with X by Baire's category theorem. Then any $x \in X \setminus (T \cdot \partial W)$ is generic with respect to W.

(ii). Observe that the sets $T \cdot \partial W$ and

$$\mathcal{T} = \bigcup_{t \in T} \bigcup_{P \in \mathcal{P}(t)} \partial P$$

are meagre. Hence, also the union $\mathcal{T} \cup (T \cdot \partial W)$ is meagre and cannot agree with X by Baire's category theorem. Then any $x \in X \setminus (\mathcal{T} \cup (T \cdot \partial W))$ is generic with respect to W and generic with respect to φ. ☐

Proposition 9.2. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with FLC cocycle φ, proper window $W \subseteq X$ and starting point $x_0 \in X$. If x_0 is generic with respect to W and generic with respect to φ then $人_{x_0}^T(W)$ is repetitive.*

Proof. Since φ is an FLC cocycle, $人_{x_0}^T(W)$ has FLC by Proposition 8.6. Recalling the definition of repetitivity we have to show that the set

$$\mathrm{Rep}(P) = \{l \in 人_{x_0}^T(W) \mid P(R, l) = P\}$$

is relatively dense in G for all $(P, R) \in \mathcal{P}(人_{x_0}^T(W))$. Fix $R > 0$ and put $B_R = \mathrm{cl}(B_R(0)) \subseteq G$. Without loss of generality assume $0 \in 人_{x_0}^T(W)$. Fix $P = P(R, 0) = 人_{x_0}^T(W) \cap B_R$. We aim to show that

$$\mathrm{Rep}_T(P) = \{t \in T \mid t \cdot x_0 \in W, P(R, \varphi_t(x_0)) = P\}$$

is relatively dense in T. In this case it follows by (D2) and (D3) that also $\mathrm{Rep}(P)$ is relatively dense in G (compare also the methods used in the proof of Proposition 8.1). Put

$$I(P) = \{s \in T \mid \varphi_s(x_0) \in P\} \text{ and } J(P) = \{s \in T \mid \varphi_s(x_0) \in B_R \setminus P\}.$$

Clearly, $I(P)$ is a finite subset of T. By (Dist), we also obtain finiteness of $J(P)$. Let $X_0 = X \setminus \partial W$ and put

- $S_1 = \{x \in X_0 \mid \forall s \in I(P) : \varphi_s(x) = \varphi_s(x_0)\}$,
- $S_2 = \{x \in X_0 \mid \forall s \in I(P) \cup J(P) : s \cdot x \in W \iff s \cdot x_0 \in W\}$
- $S_3 = \{x \in X_0 \mid \nexists s \in T \setminus (I(P) \cup J(P)) : \varphi_s(x) \in B_R\}$.

Using Lemma 8.3, for $l = \varphi_t(x_0) \in 人_{x_0}^T(W)$, we obtain

$$P(R, l) = \{\varphi_s(t \cdot x_0) \mid st \cdot x_0 \in W, \varphi_s(t \cdot x_0) \in B_R\}.$$

We observe that $t \cdot x_0 \in S_1 \cap S_2$ yields that $P \subseteq P(R, \varphi_t(x_0))$, whereas the additional condition $t \cdot x_0 \in S_3$ ensures that $P = P(R, \varphi_t(x_0))$. This shows that the inclusion

$$\{t \in T \mid t \cdot x_0 \in \mathcal{N}(P)\} \subseteq \mathrm{Rep}_T(P)$$

holds, where $\mathcal{N}(P) = S_1 \cap S_2 \cap S_3$. Further, we note that $x_0 \in \mathcal{N}(P)$ and hence $\mathcal{N}(P) \neq \emptyset$. To conclude the proof we aim to show the existence of an open subset of $\mathcal{N}(P)$.

Given $s \in T$ put $M(s) = \{x \in X_0 \mid \varphi_s(x) = \varphi_s(x_0)\}$. Then we obtain

$$S_1 = \bigcap_{s \in I(P)} M(s).$$

Since φ has (PwCo) and x_0 is generic with respect to φ, for each $s \in I(P)$ there exists an open neighbourhood $U_s(x_0) \subseteq M(s)$. Hence,

$$\bigcap_{s \in I(P)} U_s(x_0) \subseteq \bigcap_{s \in I(P)} M(s).$$

Due to finiteness of $I(P)$ we have proven the existence of an open subset of S_1.

Put

$$W_1 = \bigcap_{s \in I(P)} s^{-1} \cdot \mathrm{int}(W) \text{ and } W_2 = \bigcap_{s \in J(P)} s^{-1} \cdot (X \setminus W).$$

Then we have $S_2 = W_1 \cap W_2$. Since both W_1 and W_2 are finite intersections of open sets, the set S_2 is open.

Let $K(P) = I(P) \cup J(P)$ and put $\tilde{R} = \max_{s \in K(P)} d_G(0, \varphi_s(x_0))$. By (Dist), there exists some $\rho = \rho(\tilde{R}) > 0$ such that for all $x \in X$ and $s \in T$ holds $d_G(\varphi_s(x), 0) > \tilde{R}$ whenever we have $d_T(s, 0) > \rho$. In particular, we have $\rho \geq \max_{s \in K(P)} d_T(0, s)$. Due to genericity of φ

with respect to x_0, for each s with $d_T(0,s) \leq \rho$ we may find some small open neighbourhood $U'_s(x_0) \subseteq X$ such that for all $x \in U'_s(x_0)$ we have $\varphi_s(x) = \varphi_s(x_0)$ as well as $\varphi_u(x) \notin B_{\tilde{R}}$ for all $u \in T \setminus B_\rho$. Hence there exists no $u \in T \setminus B_\rho \subseteq T \setminus K(P)$ with $\varphi_u(x) \in B_R$ for all $x \in \bigcap_{s:d_T(0,s)\leq\rho} U'_s(x_0)$. Since T is a discrete subgroup with a proper metric, the latter intersection is finite and thus open.

Altogether, we have proven the existence of an open neighbourhood of x_0 contained in $\mathcal{N}(P)$. By minimality of (X,T), the set $\{t \in T \mid t \cdot x_0 \in \mathcal{N}(P)\}$ is syndetic. Hence, $\mathrm{Rep}(P)$ is relatively dense in G. ☐

In case of non-FLC cocycles, we obtain a similar result for almost repetitivity (compare Section 4.5).

Proposition 9.3. *Let $\mathcal{D} = (X,T,G,\varphi)$ be a DCPS with non-FLC cocycle φ, proper window $W \subseteq X$ and starting point $x_0 \in X$. If x_0 is generic with respect to W and generic with respect to φ, then $\curlywedge^T_{x_0}(W)$ is almost repetitive.*

Proof. For given $R > 0$ put $B_R = \mathrm{cl}(B_R(0)) \subseteq G$. Observe that (D1) yields that for each $R > 0$ the intersection $\curlywedge^T_{x_0}(W) \cap B_R$ is finite. As already seen in the proof of Proposition 9.2 it is sufficient to show that

$$\mathrm{Rep}_T = \{t \in T \mid t \cdot x_0 \in W, d_R(\curlywedge^T_{t\cdot x_0}(W), \curlywedge^T_{x_0}(W)) < \varepsilon\}$$

is syndetic for all $\varepsilon > 0$ and $R > 0$.

Fix some $\varepsilon > 0$ and $R > 0$. Following the notations of the FLC-case put $P = \curlywedge^T_{x_0}(W) \cap B_R$. Then define index sets

- $A_\varepsilon(P) = \{s \in T \mid \varphi_s(x_0) \in B_{R+\varepsilon}\}$,
- $I(P) = \{s \in T \mid \varphi_s(x_0) \in P\}$,
- $I_\varepsilon(P) = \{s \in A_\varepsilon(P) \mid \varphi_s(x_0) \notin B_R\}$.

We want to point out that the elements of $I_\varepsilon(P)$ correspond to points which are included in $B_{R+\varepsilon} \setminus B_R$. In particular, we have $\varphi_s(x_0) \notin P$ whenever $s \in I_\varepsilon(P)$. Let $X_0 = X \setminus \partial W$ and put

- $S_1 = \{x \in X_0 \mid \forall s \in I(P) : d_G(\varphi_s(x_0), \varphi_s(x)) < \varepsilon\}$,
- $S_2 = \{x \in X_0 \mid \forall s \in A_\varepsilon(P) : s \cdot x \in W \Leftrightarrow s \cdot x_0 \in W\}$,
- $S_3 = \{x \in X_0 \mid \nexists s \in T \setminus A_\varepsilon(P) : \varphi_s(x) \in B_{R+\varepsilon}\}$,
- $S_4 = \{x \in X_0 \mid \forall s \in I_\varepsilon(P) : d_G(\varphi_s(x_0), \varphi_s(x)) < \varepsilon\}$.

The sets S_1, S_2 and S_3 are similar to the sets defined in the proof of Proposition 9.2, whereby they contain also points whose images under φ are contained in $B_{R+\varepsilon}$. As we will see later, the latter set S_4 will be used to describe points $\varphi_s(x) \in B_R$ being ε-close to points $\varphi_s(x_0) \in B_{R+\varepsilon} \setminus B_R$. Finally, put

$$\mathcal{N}(P) = \bigcap_{i=1}^{4} S_i.$$

Now let $t \in T$ such that $t \cdot x_0 \in \mathcal{N}(P)$. Since $t \cdot x_0 \in S_1$, we have $\varphi_s(x_0) \in B_\varepsilon(\varphi_s(t \cdot x_0))$ for all $s \in I(P)$. Together with $t \cdot x_0 \in S_2$ we obtain

$$P = \curlywedge^T_{x_0}(W) \cap B_R \subseteq (\curlywedge^T_{t\cdot x_0}(W))_\varepsilon = \bigcup_{p \in \curlywedge^T_{t\cdot x_0}(W)} B_\varepsilon(p).$$

On the other hand, $t \cdot x_0 \in S_4$ implies $\varphi_s(t \cdot x_0) \in B_\varepsilon(\varphi_s(x_0))$ for all $s \in I_\varepsilon(P)$. Due to $t \cdot x_0 \in S_1$ we obtain also $\varphi_s(t \cdot x_0) \in B_\varepsilon(\varphi_s(x_0))$ for all $s \in I(P)$ with $\varphi_s(t \cdot x_0) \in B_R$. Then $t \cdot x_0 \in S_2 \cap S_3$ yields $\curlywedge^T_{t\cdot x_0}(W) \cap B_R \subseteq (\curlywedge^T_{x_0}(W))_\varepsilon$. These observations imply $t \in \mathrm{Rep}_T$ and hence the inclusion

$$\{t \in T \mid t \cdot x_0 \in \mathcal{N}(P)\} \subseteq \mathrm{Rep}_T$$

holds. To conclude the proof we need to show the existence of an open subset of $\mathcal{N}(P)$ containing x_0.

Given $s \in I(P)$ put $M(s) = \{x \in X_0 \mid \varphi_s(x) \in B_\varepsilon(\varphi_s(x_0))\}$. Clearly,

$$S_1 = \bigcap_{s \in I(P)} M(s) = \bigcap_{s \in I(P)} \varphi_s^{-1}(B_\varepsilon(\varphi_s(x_0))).$$

Using (PwCts) and genericity with respect to φ, we obtain the existence of an open subset $U_s(x_0) \subseteq M(s)$. Due to finiteness of $I(P)$, the intersection $\bigcap_{s \in I(P)} U_s(x_0)$ is an open subset of S_1 containing x_0. Analogously, we prove the existence of an open neighbourhood of x_0 contained in S_4.

Let $I_1 = \{s \in A_\varepsilon(P) \mid s \cdot x_0 \in W\}$ and $I_2 = A_\varepsilon(P) \setminus I_1$. Then we obtain $S_2 = W_1 \cap W_2$, where

$$W_1 = \bigcap_{s \in I_1} s^{-1} \cdot \operatorname{int}(W) \text{ and } W_2 = \bigcap_{s \in I_2} s^{-1} \cdot (X \setminus W).$$

Since both W_1 and W_2 are finite intersections of open sets, the set S_2 is also open.

Put $\tilde{R} = \max_{s \in A_\varepsilon(P)} d_G(0, \varphi_s(x_0))$. By (Dist), there exists some $\rho = \rho(\tilde{R}) > 0$ such that for all $x \in X$ and $s \in T$ holds $d_G(\varphi_s(x), 0) > \tilde{R}$ whenever we have $d_T(s, 0) > \rho$. In particular, we have $\rho \geq \max_{s \in A_\varepsilon(P)} d_T(0, s)$. Due to genericity of φ with respect to x_0, for each s with $d_T(0, s) \leq \rho$ we may find some small open neighbourhood $U_s'(x_0) \subseteq X$ such that for all $x \in U_s'(x_0)$ we have $d_G(\varphi_s(x_0), \varphi_s(x)) < \varepsilon$ as well as $\varphi_u(x) \notin B_{\tilde{R}}$ for all $u \in T \setminus B_\rho$. Hence there exists no $u \in T \setminus B_\rho \subseteq T \setminus A_\varepsilon(P)$ with $\varphi_u(x) \in B_R$ for all $x \in \bigcap_{s : d_T(0,s) \leq \rho} U_s'(x_0)$. Since T is a discrete subgroup with a proper metric, the latter intersection is finite and thus open.

Altogether we have shown that $\mathcal{N}(P)$ contains an open neighbourhood of x_0. By minimality of (X, T), the set $\{t \in T \mid t \cdot x_0 \in \mathcal{N}(P)\}$ is hence syndetic. Thus, Rep_T is relatively dense in G. ☐

9.2 Uniform Patch Frequencies

The goal of this section is to find a sufficient condition for dynamical model sets to have uniform patch frequencies, analogous to Lemma 4.12(ii). We start with the following lemma which kind of generalizes Lemma 3.33. Note that we provide actually two variants of this statement: the first variant holds in arbitrary locally compact abelian groups but requires more restrictive cocycle properties. The second variant provides a similar statement along balls in Euclidean space, but requires less assumptions on the cocycle.

Lemma 9.4. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS.*

(1) Assume $T \leq G$ is a lattice. Additionally, suppose φ is a strict UPF cocycle and $(A_n)_{n \in \mathbb{N}}$ is a van Hove sequence in G. Then the following holds.

(i) The sequence

$$F_n(x) = \{t \in T \mid \varphi_t(x) \in A_n\}$$

is a van Hove sequence in T for all $x \in X$.

(ii) There exists a constant $\kappa > 0$ such that for all $x \in X$ we have

$$\lim_{n \to \infty} \frac{\Theta_T(F_n(x))}{\Theta_G(A_n)} = \kappa.$$

(2) Assume $T = \mathbb{Z}^N$ and $G = \mathbb{R}^N$. Suppose φ is a UPF cocycle and let $B_n \subseteq \mathbb{R}^N$ denote the closed balls of radius n centered at the origin. Then the followings holds.

(i') The sequence

$$F_n(x) = \{t \in \mathbb{Z}^N \mid \varphi_t(x) \in B_n\}$$

is a van Hove sequence in $\mathbb{Z}^N$ for all $x \in X$.

(ii') For all $x \in X$ we have

$$\lim_{n\to\infty} \frac{\sharp(F_n(x))}{\mathrm{Leb}(B_n)} = 1.$$

Proof. Throughout the proof fix $x \in X$ and put $F_n = F_n(x)$.

(i). Clearly, $(F_n)_{n\in\mathbb{N}}$ is an increasing sequence of increasing and compact subsets of T. Since φ is a strict UPF cocycle, there exists an injective homomorphism $H : T \to G$ and a $C > 0$ such that φ decomposes into $\varphi_s(x) = H(s) + o_x(s)$, where $o_x(s) \in \mathrm{cl}(B_C(0))$ for all $x \in X$ and $s \in T$. In the following put $K = \mathrm{cl}(B_C(0))$. Observe that we have

$$F_n = \{t \in T \mid H(t) + o_x(t) \in A_n\} \subseteq \{t \in T \mid H(t) \in A_n + K\} = H^{-1}(A_n + K). \tag{9.2.1}$$

Put $C_n = \mathrm{cl}\left(A_n \setminus \partial^K A_n\right)$. Similarly as above we obtain the inclusion

$$H^{-1}(C_n) \subseteq F_n. \tag{9.2.2}$$

Note that, by Lemma 2.11 and Lemma 3.33, both $H^{-1}(A_n + K)$ as well as $H^{-1}(C_n)$ are van Hove sequences.

Let $L \subseteq T$ be compact. Without loss of generality we assume that L is a symmetric neighbourhood of $0 \in T$. Observe that we have

$$\partial^L F_n = (F_n + L) \setminus \bigcap_{g\in L} g + F_n$$

as well as

$$H^{-1}\left(\bigcap_{g\in L} g + C_n\right) \subseteq \bigcap_{g\in L} g + F_n.$$

Together with (9.2.1) and (9.2.2) the above identities yield

$$\frac{\Theta_T(\partial^L F_n)}{\Theta_T(F_n)} \le \frac{\Theta_T\left(H^{-1}(A_n) + H^{-1}(K) + L\right)}{\Theta_T\left(H^{-1}(C_n)\right)} - \frac{\Theta_T\left(H^{-1}\left(\bigcap_{g\in L} g + C_n\right)\right)}{\Theta_T\left(H^{-1}(C_n)\right)}. \tag{9.2.3}$$

In the following, we aim to show that the right side of the above equation tends to zero as n tends to infinity. Before we start, we would like to point out that we have $\Theta_T(H^{-1}(\boldsymbol{.})) = \Theta_\Gamma(\boldsymbol{.} \cap \Gamma)$, where $\Gamma = H(T)$. Due to injectivity of H, the set Γ is also a lattice in G.

Put $K' = H^{-1}(K) + L \subseteq T$. Clearly, K' is compact. Now the first term on the right side of (9.2.3) reads as

$$\frac{\Theta_T(H^{-1}(A_n) + K')}{\Theta_T(H^{-1}(C_n))} = \frac{\Theta_\Gamma((A_n + H(K')) \cap \Gamma)}{\Theta_G(A_n)} \cdot \frac{\Theta_G(A_n)}{\Theta_\Gamma(C_n \cap \Gamma)}.$$

By Lemma 2.11 and Theorem 3.31 we obtain

$$\lim_{n\to\infty} \frac{\Theta_\Gamma((A_n + H(K')) \cap \Gamma)}{\Theta_G(A_n)} = \lim_{n\to\infty} \left(\frac{\Theta_\Gamma((A_n + H(K')) \cap \Gamma)}{\Theta_G(A_n + H(K'))} \cdot \frac{\Theta_G(A_n + H(K'))}{\Theta_G(A_n)}\right) = \frac{1}{\mu(G/\Gamma)},$$

where μ denotes the unique G-invariant measure on G/Γ. Similarly, we calculate

$$\lim_{n\to\infty} \frac{\Theta_G(A_n)}{\Theta_\Gamma(C_n \cap \Gamma)} = \lim_{n\to\infty} \left(\frac{\Theta_G(A_n)}{\Theta_G(C_n)} \cdot \frac{\Theta_G(C_n)}{\Theta_\Gamma(C_n \cap \Gamma)}\right) = \mu(G/\Gamma),$$

and hence we showed

$$\lim_{n\to\infty} \frac{\Theta_T(H^{-1}(A_n) + K')}{\Theta_T(H^{-1}(C_n))} = 1.$$

Regarding the second term on the right side of (9.2.3), we have $H^{-1}\left(\bigcap_{g\in L} g + C_n\right) = H^{-1}(C_n) \setminus \partial^L H^{-1}(C_n)$. Recalling that $H^{-1}(C_n)$ is a van Hove sequence, this yields together with Lemma 2.11 that

$$\lim_{n\to\infty} \frac{\Theta_T\left(H^{-1}\left(\bigcap_{g\in L} g + C_n\right)\right)}{\Theta_T\left(H^{-1}(C_n)\right)} = \lim_{n\to\infty} \frac{\Theta_T(H^{-1}(C_n) \setminus \partial^L H^{-1}(C_n))}{\Theta_T(H^{-1}(C_n))} = 1.$$

Hence,

$$\lim_{n\to\infty}\frac{\Theta_T(\partial^L F_n)}{\Theta_T(F_n)}=0.$$

(ii). We define the values

$$L_n=\frac{\Theta_T(H^{-1}(C_n))}{\Theta_T(H^{-1}(A_n))}\text{ as well as }U_n=\frac{\Theta_T(H^{-1}(A_n+K))}{\Theta_T(H^{-1}(A_n))}.$$

Using the fact that $(C_n)_{n\in\mathbb{N}}$ is a van Hove sequence, we may calculate

$$L_n=\frac{\Theta_\Gamma(C_n\cap\Gamma)}{\Theta_G(C_n)}\cdot\frac{\Theta_G(A_n)}{\Theta_\Gamma(A_n\cap\Gamma)}\cdot\frac{\Theta_G(C_n)}{\Theta_G(A_n)}.$$

A similar identity holds for U_n, and, by the Lattice Point Counting Theorem 3.31, we obtain

$$\lim_{n\to\infty}L_n=\lim_{n\to\infty}U_n=1. \tag{9.2.4}$$

Further the identity

$$\frac{\Theta_T(F_n)}{\Theta_G(A_n)}=\frac{\Theta_T(F_n)}{\Theta_\Gamma(A_n\cap\Gamma)}\cdot\frac{\Theta_\Gamma(A_n\cap\Gamma)}{\Theta_G(A_n)}$$

holds. By the Lattice Point Counting Theorem 3.31 we now obtain

$$\lim_{n\to\infty}\frac{\Theta_\Gamma(A_n\cap\Gamma)}{\Theta_G(A_n)}=\frac{1}{\mu(G/\Gamma)}.$$

According to (9.2.1) and (9.2.2) we have

$$L_n\le\frac{\Theta_T(F_n)}{\Theta_\Gamma(A_n\cap\Gamma)}\le U_n,$$

which, together with Equation (9.2.4), yields

$$\lim_{n\to\infty}\frac{\Theta_T(F_n)}{\Theta_G(A_n)}=\frac{1}{\mu(G/\Gamma)}.$$

(i'). Fix $\varepsilon>0$. Due to (UDP) we may decompose φ into an injective homomorphism $H:\mathbb{Z}^N\to\mathbb{R}^N$ and a function $o_x(t)$ which satisfies $\|o_x(t)\|\le\varepsilon\|t\|$ for large enough $\|t\|$. Now assume that $\varphi_t(x)\in B_k$ and let $\|t\|$ be large. Then we may calculate $\|H(t)\|\le k+\varepsilon\|t\|$. Since H is a homomorphism there exists a matrix $A\in\mathbb{R}^{N\times N}$ such that $H(t)=At$. Then we obtain $\|t\|\cdot((\|At\|/\|t\|)-\varepsilon)\le k$ and therefore $\|t\|\le k/(|||A|||-\varepsilon)$. Thus, $\|o_x(t)\|\le\varepsilon k/(|||A|||-\varepsilon)$. Put $\delta=\delta(H)=|||A|||-\varepsilon$. Then we obtain $H(t)\in B_{k+k\varepsilon/\delta}=B_{k(1+\varepsilon/\delta)}$, i.e., $F_k\subseteq H^{-1}(B_{k(1+\varepsilon/\delta)})$. On the other hand, we also have $H^{-1}(B_{k(1-\varepsilon/\delta)})\subseteq F_k$.

Similar to Equation (9.2.3) the above yields

$$\frac{\sharp(\partial^L F_k)}{\sharp(F_k)}\le\frac{\sharp(H^{-1}(B_{k(1+\varepsilon/\delta)})+L)}{\sharp\left(H^{-1}(B_{k(1-\varepsilon/\delta)})\right)}-\frac{\sharp\left(H^{-1}(B_{k(1-\varepsilon/\delta)})\setminus\partial^L H^{-1}(B_{k(1-\varepsilon/\delta)})\right)}{\sharp\left(H^{-1}(B_{k(1-\varepsilon/\delta)})\right)}$$

for some compact $L\subseteq\mathbb{Z}^N$. Proceeding similar to (i) we obtain

$$\lim_{k\to\infty}\frac{\sharp(H^{-1}(B_{k(1+\varepsilon/\delta)})+L)}{\sharp\left(H^{-1}(B_{k(1-\varepsilon/\delta)})\right)}=\left(\frac{1+\frac{\varepsilon}{\delta}}{1-\frac{\varepsilon}{\delta}}\right)^N$$

and

$$\lim_{k\to\infty}\frac{\sharp\left(H^{-1}(B_{k(1-\varepsilon/\delta)})\setminus\partial^L H^{-1}(B_{k(1-\varepsilon/\delta)})\right)}{\sharp\left(H^{-1}(B_{k(1-\varepsilon/\delta)})\right)}=1.$$

Since ε was chosen arbitrarily, we obtain

$$\lim_{k\to\infty}\frac{\sharp(\partial^L F_k)}{\sharp(F_k)}=0.$$

(ii'). Observe that $\mathrm{Leb}(\mathbb{R}^N/\mathbb{Z}^N) = 1$. For fixed $\varepsilon > 0$ and large enough $k \in \mathbb{N}$ put

$$L_k = \frac{\sharp\left(H^{-1}\left(B_{k(1-\varepsilon/\delta)}\right)\right)}{\sharp(H^{-1}(B_k))} \text{ as well as } U_k = \frac{\sharp\left(H^{-1}\left(B_{k(1+\varepsilon/\delta)}\right)\right)}{\sharp(H^{-1}(B_k))}.$$

Proceeding analogously to (ii), this leads to $1 - \frac{\varepsilon}{\delta} \leq \lim_{k\to\infty} \frac{\sharp F_k}{\mathrm{Leb}(B_k)} \leq 1 + \frac{\varepsilon}{\delta}$. Since ε was chosen arbitrarily the claim follows. □

Remark 9.5. As seen in the proof above, the constant in (ii) does not depend on the choice of the van Hove sequence $(A_n)_{n\in\mathbb{N}}$. Instead, it just depends on the lattice.

This lemma gives us the necessary tools to prove the main statement of this section.

Proposition 9.6. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Suppose, the following additional claims are true:*

(i) T is a lattice in G,

(ii) X is metrizable and (X, T, μ) is uniquely ergodic,

(iii) φ is a strict UPF cocycle and each element of the corresponding partitions of X is regular.

Then the following holds: If W is regular, then $\curlywedge_{x_0}^T(W)$ has uniform patch frequencies.

Proof. During this proof, let $B_R = \mathrm{cl}(B_R(0)) \subseteq G$ denote the compact ball of radius $R > 0$ centered around the neutral element 0 of G. Furthermore, for given $t \in T$ and $K \subseteq G$ we introduce the notation

$$\varphi_t^{-1}(K) = \{x \in X \mid \varphi_t(x) \in K\}.$$

Since each UPF cocycle is an FLC cocycle, Proposition 8.6 yields finite local complexity of $\curlywedge_{x_0}^T(W)$. Suppose $(A_n)_{n\in\mathbb{N}}$ is a van Hove sequence in G. Fix some $g \in G$. For each patch $P \in \mathcal{P}(\curlywedge_{x_0}^T(W))$ define

$$\mathcal{A}_n(g, P) = \left\{ l \in (\curlywedge_{x_0}^T(W) - g) \cap A_n \mid P(R, l) = P \right\}.$$

Since d_G is G-invariant, we may assume without loss of generality that $g = 0$. Now we have to show that the limit

$$L(P) = \lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \sharp\mathcal{A}_n(g, P) \tag{9.2.5}$$

exists uniformly along $(A_n)_{n\in\mathbb{N}}$ in g for all $(P, R) \in \mathcal{P}(\curlywedge_{x_0}^T(W))$.

To that end, fix $R > 0$ and without loss of generality consider the patch $P = P(R, 0)$. Put $I(P) = \{s \in T \mid \varphi_s(x_0) \in P\}$. Due to (Dist) we may find some $\rho = \rho(R) > 0$ such that we have $I(P) \subseteq B_\rho(0) \subseteq T$. For the sake of simplicity we may assume

$$I(P) = \{s_1, \ldots, s_N\}.$$

Assume $l = \varphi_{t(l)}(x_0) = \varphi_t(x_0) \in \mathcal{A}_n(g, P)$ for some $n \in \mathbb{N}$. According to Lemma 8.3, this means that there exist $s_1', \ldots, s_N' \in T$ such that

$$\{\varphi_{s_1}(x_0), \ldots, \varphi_{s_N}(x_0)\} = \{\varphi_{s_1'}(t \cdot x_0), \ldots, \varphi_{s_N'}(t \cdot x_0)\}.$$

Note that this is equivalent to

$$t \cdot x_0 \in \mathcal{S} = \left\{ x \in X : \exists s_1', \ldots, s_N' : x \in \bigcap_{i=1}^{3} S_i(s_1', \ldots, s_N') \right\},$$

where

- $S_1(s_1', \ldots, s_N') = \{x \in X \mid \forall s_i \in I(P) : \varphi_{s_i}(x_0) = \varphi_{s_i'}(x)\}$,

- $S_2(s'_1, \dots, s'_N) = \{x \in X \mid \forall i = 1, \dots, N : \chi_W(s'_i \cdot x) = 1\}$,
- $S_3(s'_1, \dots, s'_N) = \{x \in X \mid \nexists s \in T \setminus \{s'_1, \dots, s'_N\} : \varphi_s(x) \in B_R$ and $s \cdot x \in W\}$.

Note that we always have $x_0 \in \mathcal{S}$. Now it is easy to see that we have

$$\begin{aligned}\sharp\mathcal{A}_n(g, P) &= \sharp\{t \in T \mid \varphi_t(x_0) \in A_n, t \cdot x_0 \in \mathcal{S}, t \cdot x_0 \in W\} \\ &= \sum_{t:\varphi_t(x_0)\in A_n} \chi_{\mathcal{S}\cap W}(t \cdot x_0).\end{aligned}$$

To apply Lemma 3.27 it is necessary to show $\mu(\partial(\mathcal{S} \cap W)) = 0$. To that end, recall that (X, T, μ) was supposed to be measure-preserving. Since we have

$$\mu(\partial(\mathcal{S} \cap W)) \leq \mu(\partial\mathcal{S} \cup \partial W) \leq \mu(\partial\mathcal{S}) + \mu(\partial W)$$

as well as $\mu(\partial W) = 0$, it is sufficient to show $\mu(\partial\mathcal{S}) = 0$. Thus, we are going to show

$$\mu(\partial S_i(s'_1, \dots, s'_N)) = 0$$

for arbitrary but fixed $s'_1, \dots, s'_N \in T$. For the sake of simplicity, in the following we will use the notation $S_i = S_i(s'_1, \dots, s'_N)$.

Clearly,

$$S_1 = \bigcap_{s_i\in I(P)} \{x \in X \mid \varphi_{s_i}(x_0) = \varphi_{s'_i}(x)\}$$

and hence

$$\begin{aligned}\mu(\partial S_1) &\leq \mu\left(\bigcup_{s_i\in I(P)} \partial\{x \in X \mid \varphi_{s_i}(x_0) = \varphi_{s'_i}(x)\}\right) \\ &\leq \sum_{s_i\in I(P)} \mu\left(\partial\left\{x \in X \mid \varphi_{s_i}(x_0) = \varphi_{s'_i}(x)\right\}\right).\end{aligned}$$

Observe that we have $\{x \in X \mid \varphi_{s_i}(x_0) = \varphi_{s'_i}(x)\} = \varphi^{-1}_{s'_i}(\varphi_{s_i}(x_0))$. Since φ has (PwCo), this preimage equals an element of $\mathcal{P}(s'_i)$, for which, by definition, the measure of the boundary vanishes. Hence, $\mu(\partial S_1) = 0$.

A similar estimate as in the first case shows

$$\mu(\partial S_2) \leq \sum_{i=1}^{N} \mu(\partial\{x \in X \mid \chi_W(s'_i \cdot x) = 1\}).$$

Then, due to

$$\{x \in X \mid \chi_W(s'_i \cdot x) = 1\} = \{x \in X \mid x \in (s'_i)^{-1} \cdot W\} = (s'_i)^{-1} \cdot W,$$

we obtain

$$\mu(\partial\{x \in X \mid \chi_W(s'_i \cdot x) = 1\}) = \mu(\partial((s'_i)^{-1} \cdot W)) = \mu(\partial W) = 0.$$

Put $I' = \{s'_1, \dots, s'_N\}$. Then we have

$$S_3 = \bigcap_{s\in T\setminus I'} \{x \in X \mid \varphi_s(x) \notin B_R\} \cap \{x \in X \mid \chi_W(s \cdot x) = 0\}.$$

Hence,

$$\mu(\partial S_3) \leq \sum_{s\in T\setminus I'} \mu(\partial\{x \in X \mid \varphi_s(x) \notin B_R\}) + \mu(\partial\{x \in X \mid \chi_W(s \cdot x) = 0\}).$$

Fix some $s_0 \in T \setminus I'$. For the first term of the above sum we may calculate

$$\mu(\partial\{x \in X \mid \varphi_s(x) \notin B_R\}) = \mu\left(\partial(\varphi_{s_0}^{-1}(B_R))^c\right) = \mu(\partial(\varphi_{s_0}^{-1}(B_R))) \\ = \mu(\partial\{x \in X \mid \varphi_{s_0}(x) \in B_R\}).$$

However, since we consider a fixed time s_0 and the cocycle has (PwCo), we obtain

$$\partial\{x \in X \mid \varphi_{s_0}(x) \in B_R\} \subseteq \bigcup_{j=1}^{K} \partial P_j(s_0),$$

where $K = \sharp\mathcal{P}(s_0) \in \mathbb{N}$ and $P_j(s_0) \in \mathcal{P}(s_0)$. Since all elements of $\mathcal{P}(s_0)$ are regular, we have also $\mu(\partial\{x \in X \mid \varphi_{s_0}(x) \in B_R\}) = 0$. On the other hand, we compute

$$\mu(\partial\{x \in X \mid \chi_W(s_0 \cdot x)\}) = \mu\left(\partial\left(s_0^{-1} \cdot W\right)^c\right) = \mu(\partial W) = 0.$$

Hence, we obtain $\mu(\partial S_3) = 0$ and therefore we have shown that

$$\mu(\partial(\mathcal{S} \cap W)) = 0.$$

Now let $F_n = F_n(x_0) = \{t \in T \mid \varphi_t(x_0) \in A_n\}$. We write Equation (9.2.5) as

$$L(P) = \lim_{n\to\infty} \frac{1}{\Theta_G(A_n)} \sum_{t \in F_n} \chi_{\mathcal{S}\cap W}(t \cdot x_0). \tag{9.2.6}$$

Note that, since φ is a strict UPF cocycle, $(F_n)_{n\in\mathbb{N}}$ is a van Hove sequence in T due to Lemma 9.4. We may reformulate (9.2.6) as

$$L(P) = \lim_{n\to\infty} \frac{\Theta_T(F_n)}{\Theta_G(A_n)} \cdot \frac{1}{\Theta_T(F_n)} \sum_{t \in F_n} \chi_{\mathcal{S}\cap W}(t \cdot x_0).$$

Now Lemma 9.4(ii) yields the existence of some $\kappa > 0$ (which only depends on the lattice T) such that $\lim_{n\to\infty} \Theta_T(F_n)/\Theta_G(A_n) = \kappa$. Recalling that $\mu(\partial(\mathcal{S} \cap W)) = 0$, Lemma 3.27 then yields

$$L(P) = \kappa \cdot \mu(\mathcal{S} \cap W),$$

where the convergence is uniform in $t = t(l)$. Hence, $\curlywedge_{x_0}^{T}(W)$ has uniform patch frequencies. ☐

Remark 9.7. As already mentioned in Remark 9.5, the constant in Lemma 9.4(ii) does not depend on the choice of the concrete van Hove sequence. Likewise, the limit in Equation 9.2.6 is independent of the choice of the van Hove sequence as well. Hence, if $\curlywedge_{x_0}^{T}(W)$ has UPF, it has UPF along every van Hove sequence in G. Moreover, the limit is independent of the choice of the sequence.

As an immediate consequence of Lemma 9.4 and the previous proof we obtain

Corollary 9.8. *Let $\mathcal{D} = (X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Suppose, the following additional claims are true:*

(i) T is a lattice in G,

(ii) X is metrizable and (X, T, μ) is uniquely ergodic,

(iii) φ is a UPF cocycle and each element of the corresponding partitions of X is regular,

Then the following holds: If W is regular, then $\curlywedge_{x_0}^{T}(W)$ has uniform patch frequencies along $(\mathrm{cl}(B_n(0)))_{n\in\mathbb{N}}$.

9.3 Bounded Remainder Sets and the Meyer Property

We consider a DCPS $\mathcal{D} = (X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ with proper window $W \subseteq X$ and starting point $x_0 \in X$. In this section, we want to point out a crucial connection between the dynamical system $(X, \mathbb{Z}^N)$ and the local structure of the corresponding model set. It turns out that we have the following interplay: the more restrictive the dynamics on X are, the more restrictive the local structure of the model set is (and vice versa).

In this section, we will consider the following setup. Let $(X, \mathbb{Z}^N)$ be a minimal dynamical system. As discussed in Section 3.3, we might represent the $\mathbb{Z}^N$-action as

$$H : \mathbb{Z}^N \times X \to X : x \mapsto H^{(n_1,\dots,n_N)}(x). \tag{9.3.1}$$

We assume that we have $H^{(n_1,\dots,n_N)} = H_1^{n_1} \circ \dots \circ H_N^{n_N}$ and that the actions commute. Further, we introduce N functions

$$f_i : X \to \mathbb{R}^N : x \mapsto \sum_{j=1}^{K_i} a_j^i \chi_{X_j^i}(x), \tag{9.3.2}$$

where each family of sets $\mathcal{X}^i = \{X_j^i \mid j = 1, \dots, K_i\}$ is a partition of X and the coefficients are chosen such that $a_j^i \in \mathbb{R}^N_{\geq 0}$ as well as $a_j^i \neq 0$ for all $i, j \in \{1, \dots, N\}$. Additionally, we assume that

$$f_i(H_j(x)) + f_j(x) = f_i(x) + f_j(H_i(x))$$

for all $x \in X$ and $i, j \in \{1, \dots, N\}$.

According to our examples in Section 8.1, we might construct a cocycle

$$\varphi : \mathbb{Z}^N \times X \to \mathbb{R}^N$$

which is generated by the N functions $f_1, \dots, f_N$. Recall that the explicit form of φ is given in Equation (8.1.2).

We want to focus on the question which additional assumptions on $(X, \mathbb{Z}^N)$ and the coefficients a_j^i are required such that a dynamical model set arising from the DCPS $(X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ satisfies certain geometrical properties like the Meyer property.

As a first approach to deal with this question we restrict the coefficients a_j^i to consist of rational numbers instead of real numbers. Then we obtain the following

Lemma 9.9. *Let $\mathcal{D} = (X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. For each $i \in \{1, \dots, N\}$ let $\mathcal{X} = (X_j^i)_{j=1}^{K_i}$ be a partition of X. Let φ be defined as in (8.1.2), where the generating functions have the form of (9.3.2) with coefficients $a_j^i \in \mathbb{Q}^N$ such that $a_j^i \neq -a_{j'}^{i'}$ for all $i, i', j, j' \in \{1, \dots, N\}$.*

Then $人_{x_0}^{\mathbb{Z}^N}(W)$ is a Meyer set.

Proof. Put $人 = 人_{x_0}^{\mathbb{Z}^N}(W)$. Since φ is an FLC cocycle and W is proper, Proposition 8.6 yields that $人$ is itself a Delone set with FLC.

Let $p \in 人$. Then there exists $\boldsymbol{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$ such that

$$\begin{aligned} p = \varphi_{\boldsymbol{n}}(x_0) &= \sum_{i=1}^{N} \left(\sum_{k=0}^{n_i - 1} \left(\sum_{j=1}^{K_i} a_j^i \chi_{X_j^i} \left(H_i^k \circ H_{i+1}^{n_{i+1}} \circ \dots \circ H_N^{n_N}(x_0) \right) \right) \right) \\ &= \sum_{i=1}^{N} \left(\sum_{j=1}^{K_i} k_j^i a_j^i \right) \end{aligned}$$

for integers $k_j^i \in \mathbb{Z}$. With the last identity it is not hard to see that

$$人 - 人 - (人 - 人) = \left\{ \sum_{i=1}^{N} \left(\sum_{j=1}^{K_i} k_j^i a_j^i \right) : k_j^i \in \mathbb{Z} \right\}.$$

Now we supposed that all vectors a_j^i consist of rational numbers. Hence, we obtain

$$0 \notin \mathrm{cl}((⅄ - ⅄ - (⅄ - ⅄)) \setminus \{0\}).$$

This immediately yields that ⅄ is a Meyer set. □

In the above lemma, the assumption of rational coefficients was crucial. The last step of the above proof won't work anymore if the coefficients consist of rationally independent numbers.

However, it is also possible to obtain Meyer sets in such a case. Indeed, as we will see in Section 11.2, this is in particular the case for dynamical CPS arising from classical CPS.

Before we discuss the case of arbitrary coefficients, we are going to introduce some new notation. First, we will discuss the case $T = \mathbb{Z}$ and $G = \mathbb{R}$, later we will generalize this to arbitrary dimensions of the acting group and the physical space. As usual we will represent the $\mathbb{Z}$-action on a compact topological space X via an homeomorphism $H : X \to X$ and write also (X, H) instead of $(X, \mathbb{Z})$. Further, we assume that $(X, \mathbb{Z})$ carries exactly one ergodic measure denoted by μ.

Consider a subset $B \subseteq X$. We say B is a *bounded remainder set (with respect to $x_0 \in X$)* if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $x \in \mathcal{O}_{\mathbb{Z}}(x_0)$ holds

$$\left| \sum_{i=0}^{n-1} \chi_B(H^i(x)) - n\mu(B) \right| \leq C, \tag{9.3.3}$$

as well as

$$\left| \sum_{i=n}^{-1} \chi_B(H^i(x)) - |n|\mu(B) \right| \leq C,$$

for all $n \in -\mathbb{N}$.

Let $\mathcal{X} = \{X_j \mid j = 1, \ldots, K\}$ denote a finite partition of X. We say $\mathcal{X}$ *consists of bounded remainder sets (with respect to $x_0 \in X$)* if each $X_j \in \mathcal{X}$ is a bounded remainder set (with respect to $x_0 \in X$). If no confusion arises we will omit to mention the dependence on x_0.

Remark 9.10. Given a finite partition $(X_j)_{j=1}^K$ which consists of bounded remainder sets, it is easy to see that the constant $C > 0$ can be chosen such that (9.3.3) is satisfied for all $j = 1, \ldots, K$ with respect to this constant.

In the following, we want to discuss examples of systems admitting bounded remainder sets. To that end, we consider the special case $X = \mathbb{S} = \mathbb{R}/m\mathbb{Z}$, $m \in \mathbb{N}$. Suppose there is a $\mathbb{Z}$-action on $\mathbb{S}$ given by

$$H : \mathbb{S} \to \mathbb{S} : x \mapsto x + \alpha \mod \mathbb{S}$$

with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. As discussed in Section 3.3, $(\mathbb{S}, H)$ is a minimal dynamical system carrying a unique ergodic measure given by the Lebesgue measure Leb on $\mathbb{S}$. Furthermore, let $\mathcal{X} = \{[0, 1-\alpha), [1-\alpha, 1)\}$ be a partition of $\mathbb{S}$. We will refer to an irrational rotation with such a partition X as above as *Sturmian rotation*.

In this situation it was shown in [Ost27] that intervals with length $\mathbb{Z} + \alpha\mathbb{Z}$ then satisfy (9.3.3). On the other hand, as shown in [Kes66], an interval is a bounded remainder set only if the length of the interval is in $\mathbb{Z} + \alpha\mathbb{Z}$. Altogether, this shows that the partition elements of Sturmian rotations are, in a certain sense, the prototype of bounded remainder sets:

Proposition 9.11 ([Ost27],[Kes66], [GL15]). *Let $X = \mathbb{R}/m\mathbb{Z}$, $m \in \mathbb{N}$. Consider the dynamical system (X, H), where $H : X \to X : x \mapsto x + \alpha \mod X$. Let $I \subseteq X$ be an interval. Then I satisfies (9.3.3) with respect to all $x \in X$ if and only if $\mathrm{Leb}(I) \in \mathbb{Z} + \alpha\mathbb{Z}$.*

Corollary 9.12. *The partition elements of a Sturmian rotation are bounded remainder sets.*

It seems natural to ask for a similar characterization of bounded remainder sets in the case of $T^M = \mathbb{R}^M/\mathbb{Z}^M$ with $M \geq 2$. We recall (9.3.1) and observe that a minimal $\mathbb{Z}$-action on $\mathbb{T}^M$ is given by

$$H : \mathbb{T}^M \to \mathbb{T}^M : x \mapsto x + \alpha \mod \mathbb{T}^M,$$

where $\alpha = (\alpha_1, \ldots, \alpha_M) \in \mathbb{R}^M$ is chosen such that all entries of α are rationally independent.

Not surprising, it is much harder to characterize bounded remainder sets on the higher dimensional torus. Despite this, there are still some results which describe such sets. As in the one-dimensional case, properties of bounded remainder sets crucially depend on the corresponding rotation vector. In the following, we want to mention a few results concerning bounded remainder sets in $\mathbb{T}^M$.

To that end, we want to briefly introduce two new notions related to the structure of higher dimensional bounded remainder sets. Given M vectors $v_1, \ldots, v_M \in \mathbb{T}^M$ we will call $P(v_1, \ldots, v_M) = \left\{\sum_{k=1}^M t_k v_k : t_k \in [0,1)\right\}$ *parallelelepiped spanned by* $v_1, \ldots, v_M$. Further, we say two measurable sets $A, B \subseteq \mathbb{T}^M$ are *equidecomposable*, if the set A can be partitioned into finitely many measurable subsets which can be reassembled by rigid motions to form a partition of B up to a set of measure zero.

Proposition 9.13 ([GL15]). *Consider the dynamical system $(\mathbb{T}^M, H)$ given as in (9.3).*

(i) Suppose $S \subseteq \mathbb{T}^M$ is a bounded remainder set. Then $\mathrm{Leb}(S) = n_0 + \sum_{i=1}^M n_i \alpha_i$, where $n_i \in \mathbb{Z}$.

(ii) Suppose $v_1, \ldots, v_M \in \mathbb{Z}\alpha + \mathbb{Z}^M$. Then $P(v_1, \ldots, v_M)$ is a bounded remainder set.

(iii) Suppose $S \subseteq \mathbb{T}^M$ is Riemann measurable. Then S is a bounded remainder set if and only if it is equidecomposable to a parallelelepiped spanned by vectors $v_1, \ldots, v_M \in \mathbb{Z}\alpha + \mathbb{Z}^M$ using translations by vectors in $\mathbb{Z}\alpha + \mathbb{Z}^M$.

Now we are going back to dynamical CPS. Recalling Lemma 4.8, every model set emerging from a dynamical CPS is a Meyer set. However, due to the larger flexibility of dynamical CPS, we can not expect this to hold anymore for arbitrary DCPS.

Our goal is to provide a method to characterize dynamical model sets which satisfy FLC or the Meyer property. To that end, consider the DCPS $(X, \mathbb{Z}, \mathbb{R}, \varphi)$ and assume that the cocycle is an FLC cocycle. Since $T = \mathbb{Z}$ and $G = \mathbb{R}$, the discussion in Section 8.1 yields that any cocycle $\varphi : \mathbb{Z} \times X \to \mathbb{R}$ is induced by a function $f : X \to \mathbb{R}$. To ensure (PwCo), we assume that φ is induced by a step function as given in (9.3.2).

As we will see in the next theorem, if the partition elements of $\mathcal{X}$ satisfy additional requirements, dynamical model sets may carry the Meyer property although the coefficients of the inducing function were chosen (almost) arbitrary.

Theorem 9.14. *Let $\mathcal{D} = (X, H, \mathbb{R}, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Assume (X, H) carries a unique ergodic measure λ. Further, let $\mathcal{X} = (X_j)_{j=1}^K$ be a partition of X and φ be the induced cocycle of a function as defined in (9.3.2). Then the following is true: If $\mathcal{X}$ consists of bounded remainder sets, then $⅄_{x_0}^{\mathbb{Z}}(W)$ is a Meyer set.*

Proof. Put $X = X$ and $⅄ = ⅄_{x_0}^{\mathbb{Z}}(X)$. Let $p \in ⅄ - ⅄$, i.e. there exist $n, m \in \mathbb{Z}$ such that

$$p = \varphi_n(x_0) - \varphi_m(x_0) = \sum_{i=0}^{n-1} f(H^i(x_0)) - \sum_{i=0}^{m-1} f(H^i(x_0)).$$

Without loss of generality we assume that $n > m > 0$. Then we have

$$p = \sum_{i=m}^{n-1} f(H^i(x_0)) = \sum_{i=m}^{n-1} \sum_{j=1}^{K} a_j \chi_{X_j}(H^i(x_0)) = \sum_{j=1}^{K} k_j a_j$$

for integers $k_j := k_j(n,m) \in \mathbb{Z}$. Now fix some $j \in \{1,\dots,K\}$. Since $\mathcal{X}$ consists of bounded remainder sets, there exists $C > 0$ such that we have

$$\left|\sum_{i=m}^{n-1} \chi_{X_j}(H^i(x_0)) - (n-m)\lambda(X_j)\right| = \left|\sum_{i=0}^{n-m-1} \chi_{X_j}(H^i(H^m(x_0))) - (n-m)\lambda(X_j)\right| \le C.$$

This leads to the following calculation:

$$\begin{aligned}\left|\sum_{i=m}^{n-1} f(H^i(x_0)) - (n-m)\int_X f \,\mathrm{d}\lambda\right| &= \left|\sum_{j=1}^{K} a_j \left(\sum_{i=m}^{n-1} \chi_{X_j}(H^i(x_0)) - (n-m)\int_X \chi_{X_j} \,\mathrm{d}\lambda\right)\right| \\ &\le \left|\sum_{j=1}^{K} a_j\right| \left|\sum_{i=m}^{n-1} \chi_{X_j}(H^i(x_0)) - (n-m)\lambda(X_j)\right| \\ &\le \left|\sum_{j=1}^{K} a_j\right| C =: \kappa.\end{aligned}$$

We aim to show that $0 \notin \mathrm{cl}(\text{人} - \text{人} - (\text{人} - \text{人}))$ in order to proof that 人 is Meyer. To that end, choose another point

$$p' = \sum_{i=m'}^{n'-1} f(H^i(x_0)) = \sum_{j=1}^{K} k'_j a_j \in (\text{人} - \text{人}) \setminus \{p\}$$

and assume that $\|p - p'\| < 1$. Note that κ does neither depend on $n - m$ nor on $n' - m'$. Hence,

$$\begin{aligned}\left||p-p'| - |n-m-(n'-m')|\int_X f \,\mathrm{d}\lambda\right| &\le \left|p - (n-m)\int_X f \,\mathrm{d}\lambda\right| + \left|p' - (n'-m')\int_X f \,\mathrm{d}\lambda\right| \\ &\le \kappa + \kappa = 2\kappa.\end{aligned}$$

As a direct consequence we obtain

$$|p-p'| - 2\kappa \le |n-m-(n'-m')| \left|\int_X f \,\mathrm{d}\lambda\right| \le |p-p'| + 2\kappa.$$

Together with $|p - p'| < 1$ this yields

$$|n-m-(n'-m')| < \frac{1+2\kappa}{\left|\int_X f \,\mathrm{d}\lambda\right|} =: \zeta. \tag{9.3.4}$$

Recall that for each $j \in \{1,\dots,K\}$ we have $\sum_{i=0}^{n-m-1} \chi_{X_j}(H^i(x_0)) = k_j$. Since $\mathcal{X}$ consists of bounded remainder sets we obtain

$$k_j \le (n-m)\lambda(X_j) + C \text{ as well as } k'_j \le (n'-m')\lambda(X_j) + C$$

for all $j \in \{1,\dots,K\}$. This leads to

$$\begin{aligned}\left||k_j - k'_j| - |n-m-(n'-m')|\lambda(X_j)\right| &\le |k_j - (n-m)\lambda(X_j)| + \left|k'_j - (n'-m')\lambda(X_j)\right| \\ &\le C + C = 2C\end{aligned}$$

and hence, by Equation (9.3.4) and $\lambda(X_j)/\left|\int_X f \,\mathrm{d}\lambda\right| < 1$,

$$|k_j - k'_j| \le 2C + |n-m-(n'-m')|\lambda(X_j) \le 2C + \zeta\lambda(X_j) < 1 + 2\kappa + 2C.$$

Thus, there exists a uniform boundary for the value of all differences of coefficients $k_j - k'_j$ occurring in the difference $p - p' = \sum_{j=1}^{K}(k_j - k'_j)a_j$ for all $p - p' \in (\text{人} - \text{人} - (\text{人} - \text{人})) \cap B_1(0)$. This means there occur only finitely many elements in $(\text{人} - \text{人} - (\text{人} - \text{人})) \cap B_1(0)$, i.e., $0 \notin \mathrm{cl}(\text{人} - \text{人} - (\text{人} - \text{人}))$. Hence, 人 is a Meyer set.

To conclude the proof, observe that each relatively dense subset of a Meyer set is itself Meyer. Since $\mathrm{int}(W) \neq \emptyset$ and $\text{人}_{x_0}^{\mathbb{Z}}(W) \subseteq \text{人}$, we have shown that $\text{人}_{x_0}^{\mathbb{Z}}(W)$ has the Meyer property. $\square$

Remark 9.15. Recalling the setting of Lemma 8.9, it is easy to see that cocycles induced by step functions defined on bounded remainder sets are SUPF cocycles. The proof shows that the compact set required for (SUDP) is given by $\mathrm{cl}(B_\kappa(0))$. Furthermore, the required homomorphism is then given by

$$n \mapsto n \cdot \int_X f \,\mathrm{d}\lambda.$$

Finally, we are going to discuss the case $\mathcal{D} = (X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$. Here, we assume that the $\mathbb{Z}^N$-action is given as in (9.3.1) and the corresponding cocycle $\varphi : \mathbb{Z}^N \times X \to \mathbb{R}^N$ is generated by N functions given as in (9.3.2). Recall that to each $i \in \{1, \ldots, N\}$ we associate a finite partition $\mathcal{X}^i$ of X. For given $\boldsymbol{n} \in \mathbb{Z}^N$ put

$$\kappa_j^i(\boldsymbol{n}) = \sum_{k=0}^{n_i - 1} \chi_{X_j^i}(H_i^k \circ H_{i+1}^{n_{i+1}} \circ \ldots \circ H_N^{n_N}(x_0)) \in \mathbb{Z}.$$

Now observe that each point $p \in \curlywedge_{x_0}^{\mathbb{Z}^N}(X) - \curlywedge_{x_0}^{\mathbb{Z}^N}(X)$ has a representation as

$$p = \varphi_{\boldsymbol{n}}(x_0) - \varphi_{\boldsymbol{m}}(x_0) = \sum_{i=1}^{N} \left(\sum_{j=1}^{K_i} a_j^i (\kappa_j^i(\boldsymbol{n}) - \kappa_j^i(\boldsymbol{m})) \right) = \sum_{i=1}^{N} \left(\sum_{j=1}^{K_i} a_j^i \kappa_j^i \right),$$

where $\kappa_j^i := \kappa_j^i(\boldsymbol{n}, \boldsymbol{m}) = \kappa_j^i(\boldsymbol{n}) - \kappa_j^i(\boldsymbol{m})$. Following the methods used in the previous proof of Theorem 9.14 we then obtain

Corollary 9.16. *Let $\mathcal{D} = (X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Assume the $\mathbb{Z}^N$-action on X is given as in (9.3.1) and the system carries a unique ergodic measure λ. For each $i \in \{1, \ldots, N\}$ let $\mathcal{X}^i = (X_j^i)_{j=1}^{K_i}$ be a finite partition of X and assume φ is a cocycle generated by N functions as given in (9.3.2). Then the following is true: If $\mathcal{X}^i$ consists of bounded remainder sets for all $i \in \{1, \ldots, N\}$, then $\curlywedge_{x_0}^{\mathbb{Z}^N}(W)$ is a Meyer set.*

Concluding this section, we want to provide an example of a Delone set which is not Meyer arising from a DCPS such that the condition of $\mathcal{X}$ consisting of bounded remainder sets is not satisfied.

To that end, consider the subsets $X_1 = [0, 1/2)$ and $X_2 = [1/2, 1)$ of the circle $\mathbb{S}$. Clearly, $\mathcal{X} = \{X_1, X_2\}$ is a partition of $\mathbb{S}$. Fix $\alpha \in [0, 1] \setminus \mathbb{Q}$ and put

$$H : \mathbb{S} \to \mathbb{S} : x \mapsto x + \alpha \mod 1.$$

By Proposition 9.11, both X_1 and X_2 are not bounded remainder sets, respectively. For that reason we will refer to the dynamical system $(\mathbb{S}, H)$ with partition $\mathcal{X}$ as *pseudo Sturmian rotation*.

Before we investigate the connection between the pseudo Sturmian rotation and dynamical model sets we first want to discuss a few consequences of X_1 and X_2 not being bounded remainder sets. Fix $j \in \{1, 2\}$. First, X_j being not a bounded remainder set means, that for all $C > 0$ there exists some $N = N(C) \in \mathbb{N}$ and $x = x(C) \in \mathcal{O}(x_0)$ such that we have

$$\left| \sum_{i=0}^{N-1} \chi_{X_j}(H^i(x)) - \frac{N}{2} \right| > C. \tag{9.3.5}$$

Note that this definition naturally extends to integer N (compare Remark 9.10(i)). However, to avoid unnecessary case analysis, we will stick to the case of N being positive in the following discussion.

For given $N \in \mathbb{N}$ and $x \in \mathcal{O}(x_0)$ put

$$k_j = k_j(N, x) = \sum_{i=0}^{N-1} \chi_{X_j}(H^i(x)).$$

The following lemma mentions a few immediate consequences of (9.3.5).

Lemma 9.17. *Fix $j \in \{1,2\}$. Then the following holds.*

(i) Let $C > 0$, $N \in \mathbb{N}$ and $x \in \mathcal{O}(x_0)$. If $k_1(N,x) - \frac{N}{2} > C$, then $k_2(N,x) - \frac{N}{2} < -C$ and vice versa. In particular we have $|k_1(N,x) - k_2(N,x)| > 2C$.

(ii) For all $x \in \mathcal{O}(x_0)$ and for all $C > 0$ there exists some $N \in \mathbb{N}$ such that we have $\left|k_j(N,x) - \frac{N}{2}\right| > C$. In particular, there exist infinitely many such N.

(iii) For each $C > 0$ and $N \in \mathbb{N}$ there exist infinitely many $x \in \mathcal{O}(x_0)$ such that we have $\left|k_j(N,x) - \frac{N}{2}\right| \leq C$.

Proof. (i) and (ii) are immediate consequences of the definition of k_j and (9.3.5), respectively.

To see (iii), fix $C > 0$ and $N \in \mathbb{N}$. Assume we have $\left|k_j(N,x) - \frac{N}{2}\right| > C$ for all $x \in \mathcal{O}(x_0)$. Without loss of generality we may assume that $k_1(N,x) - \frac{N}{2} > C$. Then we obtain

$$\int_{\mathbb{S}} k_1(N,x) - \frac{N}{2}\, \mathrm{d}x = \frac{N}{2} - \frac{N}{2} = 0 > C$$

which contradicts $C > 0$. Hence, there has to exist some $x' \in \mathcal{O}(x_0)$ satisfying the desired statement. Since $(\mathbb{S}, H)$ is an irrational rotation, we may find some small neighbourhood U of x' such that for all $x \in U \cap \mathcal{O}(x_0)$ we have $\left|k_j(N,x) - \frac{N}{2}\right| \leq C$. Minimality of $(\mathbb{S}, H)$ now ensures $\sharp(U \cap \mathcal{O}(x_0)) = \infty$. ☐

In the following we will discuss an example class of Delone sets not satisfying the Meyer property if they are arising from step functions whose corresponding partition does not consist of bounded remainder sets.

Proposition 9.18. *Let $(\mathbb{S}, H, \mathbb{R}, \varphi)$ be a DCPS with starting point $x_0 \in \mathbb{S}$. Assume that $(\mathbb{S}, H)$ is a pseudo Sturmian rotation with partition $\{X_1, X_2\} = \{[0,1/2), [1/2,1)\}$ and the cocycle φ is induced by a function*

$$f : \mathbb{S} \to \mathbb{R} : x \mapsto a_1 \chi_{X_1}(x) + a_2 \chi_{X_2}(x),$$

where a_1 and a_2 are rationally independent positive numbers. Then $人^{\mathbb{Z}}_{x_0}(\mathbb{S})$ is not a Meyer set.

Proof. Put $人 = 人^{\mathbb{Z}}_{x_0}(\mathbb{S})$. In the following we want to show that for each pair $(\alpha, \beta) \in \mathbb{N} \times (-\mathbb{N})$ we may find points $p, p' \in 人 - 人$ with $p = k_1 a_1 + k_2 a_2$ and $p' = k_1' a_1 + k_2' a_2$ with $k_1 - k_1' = \alpha$ and $k_2 - k_2' = \beta$. Together with rational independence of a_1 and a_2 this immediately yields that $人$ is not Meyer.

Note that (9.3.5) immediately yields $\left|k_j - \frac{N}{2}\right| \in \mathbb{N} + \frac{1}{2}\mathbb{N}$ for all $N \in \mathbb{N}$ and $j \in \{1,2\}$. Thus, fix $C \in \mathbb{N} + \frac{1}{2}\mathbb{N}$ and assume without loss of generality $C > 2$. Then there exists $l \in \mathbb{N}$ and $x \in \mathcal{O}(x_0)$ such that $\left|k_j(l,x) - \frac{l}{2}\right| = C$. Since $x \in \mathcal{O}(x_0)$ we may find some $m \in \mathbb{N}$ such that $x = H^m(x_0)$. Further, there exists $n \in \mathbb{N}$ satisfying $l = n - m$. We may assume that

$$k_1(l,x) - \frac{n-m}{2} = C \text{ and } k_2(l,x) - \frac{n-m}{2} = -C. \tag{9.3.6}$$

In this case we have $k_1(l,x) > k_2(l,x)$. Moreover, the coefficients $k_1(l,x)$ and $k_2(l,x)$ correspond to a point

$$\varphi_l(x) = \varphi_{n-m}(H^m(x_0)) = \varphi_n(x_0) - \varphi_m(x_0) \in 人 - 人.$$

By Lemma 9.17(iii) we may find infinitely many $m' \in \mathbb{N}$ such that we have

$$k_1(l,x') - \frac{l}{2} \leq \frac{C}{2} \text{ and } k_2(l,x') - \frac{l}{2} \geq -\frac{C}{2}, \tag{9.3.7}$$

where $x' = H^{m'}(x_0)$. As above, these coefficients correspond to a point

$$\varphi_l(x') = \varphi_{n'}(x_0) - \varphi_{m'}(x_0) \in 人 - 人$$

for some $n' \in \mathbb{N}$ with $n' - m' = l$.

This leads to

$$k_1(l,x) - k_1(l,x') \geq \frac{C}{2} > 0 \text{ as well as } k_2(l,x) - k_2(l,x') \leq -\frac{C}{2} < 0. \tag{9.3.8}$$

Furthermore, observe that, by the definition of $k_j(.,.)$, we have that one of the following holds: either we have both

$$k_1(l-1,x) = k_1(l,x) - 1 \text{ and } k_2(l-1,x) = k_2(l,x)$$

or both

$$k_1(l-1,x) = k_1(l,x) \text{ and } k_2(l-1,x) = k_2(l,x) - 1.$$

In the remaining proof we will refer to this condition as $k_j(.,.)$ being *monotone*.

In the following we fix $x \in \mathcal{O}(x_0)$ associated to (9.3.6) and an $m' \in \mathbb{N}$ (and thus $x' = H^{m'}(x_0)$) associated to (9.3.7). Monotonicity of $k_j(.,.)$ together with Equation (9.3.8) and Lemma 9.17(ii) yield that we may choose a constant $\kappa_1 = \kappa_1(l) \in \mathbb{N}$ as the smallest natural number such that $k_1(l,x) - k_1(l+\kappa_1,x') = 0$ and $k_1(l,x) - k_1(l',x') > 0$ for all $l' \in (l, l+\kappa_1) \cap \mathbb{N}$. Similarly, we may choose $\kappa_2 = \kappa_2(l) \in \mathbb{N}$ as the smallest natural number such that $k_2(l,x) - k_2(l-\kappa_2,x') = 0$ and $k_2(l,x) - k_2(l',x') < 0$ for all $l' \in (l-\kappa_2, l) \cap \mathbb{N}$.

Now suppose $l' \in (l, l+\kappa_1) \cap \mathbb{N}$ with $k_1(l,x) - k_1(l',x') > 0$. Then we immediately obtain

$$k_2(l,x) - k_2(l',x') = l - l' + k_1(l',x') - k_1(l,x) < 0.$$

Similarly, for $l' \in (l-\kappa_2, l) \cap \mathbb{N}$ with $k_2(l,x) - k_2(l',x') < 0$, we obtain $k_1(l,x) - k_1(l',x') > 0$. This yields

- $k_1(l,x) - k_1(l+\kappa_1,x') = k_2(l,x) - k_2(l-\kappa_2,x') = 0$ and $k_1(l,x) - k_1(l-\kappa_2,x') = \kappa_2$ as well as $k_2(l,x) - k_2(l+\kappa_1,x') = -\kappa_1$,
- $k_1(l,x) - k_1(l',x') > 0$ and $k_2(l,x) - k_2(l',x') < 0$ for all $l \in (l-\kappa_2, l+\kappa_1) \cap \mathbb{N}$.

By putting $\alpha_i = k_1(l,x) - k_1(l-\kappa_2+i,x')$ and $\beta_i = k_2(l,x) - k_2(l-\kappa_2+i,x')$, we obtain a finite sequence of pairs $(\alpha_i,\beta_i)_{i=0}^M \in \mathbb{N} \times (-\mathbb{N})$ such that $(\alpha_0,\beta_0) = (\kappa_2, 0)$, $(\alpha_M,\beta_M) = (0, -\kappa_1)$ and either both $\alpha_{i-1} = \alpha_i + 1$ and $\beta_{i-1} = \beta_i$ or both $\alpha_{i-1} = \alpha_i$ and $\beta_{i-1} = \beta_i + 1$. In the next step, we want to investigate further the properties of the sequence (α_i, β_i).

To that end, put $\alpha = \alpha_{i_0} = k_1(l,x) - k_1(l,x')$ and $\beta = \beta_{i_0} = k_2(l,x) - k_2(l,x')$. Clearly, $\alpha \in [0, -\kappa_2] \cap \mathbb{N}$ and $\beta \in [0, \kappa_1] \cap \mathbb{N}$. Recall, that by (9.3.8) we have $\alpha \geq C/2$ as well as $\beta \leq -C/2$. Now it is not hard to see that for all $i \in \{i_0, \ldots, M\}$ with $\alpha_i \leq \alpha$ we then have $\beta_i \leq \beta \leq -C/2$. Similarly, for all $i \in \{0, \ldots, i_0\}$ with $\beta_i \geq \beta$ we have $\alpha_i \geq \alpha \geq C/2$. Hence, as C increases, also α_0 and $|\beta_M|$ (and by monotonicity also the length of the sequence) increase.

Now for each $l' \in (l-\kappa_2, l+\kappa_1) \cap \mathbb{N}$ we may define some $C' = C'(l') > 0$ such that $k_1(l',x') - k_2(l',x') = 2C'$. Without loss of generality we may assume that $C > C'$ for all $l' \in (l-\kappa_2, l+\kappa_1) \cap \mathbb{N}$. Then we always have

$$(k_1(l,x) - k_1(l',x')) - (k_2(l,x) - k_2(l',x')) = 2(C - C').$$

By Lemma 9.17(ii) there exist infinitely many indices $\widetilde{l} \in \mathbb{N}$ satisfying (9.3.6). We denote the set of all such points by $\mathcal{S}$. For each such point in $\mathcal{S}$ we may do the same construction as above (with respect to the same point $x' \in \mathcal{O}(x_0)$). Note that we have $\kappa_j = \kappa_j(l) = \kappa_j(\widetilde{l})$ for each $\widetilde{l} \in \mathcal{S}$, $j \in \{1,2\}$. Now minimality of $(\mathbb{S}, H)$ ensures the existence of points $\widetilde{l} \in \mathcal{S}$ such that $C'(l') \neq C'(\widetilde{l'})$ for $l' \in (l-\kappa_2, l+\kappa_1) \cap \mathbb{N}$ and $\widetilde{l'} \in (\widetilde{l}-\kappa_2, \widetilde{l}+\kappa_1) \cap \mathbb{N}$ with $l' - l = \widetilde{l'} - \widetilde{l}$.

Summarizing the preceeding discussion, to each l we may associate a finite sequence $(\alpha_i,\beta_i)_{i=1}^M \subseteq \mathbb{N} \times (-\mathbb{N})$ such that we have $\alpha_0 \geq \frac{C}{2}, \beta_0 = 0, \alpha_M = 0$ and $\beta_M \leq -\frac{C}{2}$ as well as either $\alpha_{i-1} = \alpha_i + 1$ and $\beta_{i-1} = \beta_i$ or $\alpha_{i-1} = \alpha_i$ and $\beta_{i-1} = \beta_i + 1$.

Further, each different $\widetilde{l}$ satisfying (9.3.6) yields another such subset $(\gamma_i, \delta_i)_{i=0}^{M}$ such that we have $\alpha_0 = \gamma_0$, $\beta_0 = \delta_0$, $\alpha_M = \gamma_M$ and $\beta_M = \delta_M$ as well as $\alpha_i \neq \gamma_i$ and $\beta_i \neq \delta_i$ for certain indices $i \in \{1, \ldots, M\}$. In particular, all possible combinations of pairs are realized by varying over l and m'.

Since C was chosen arbitrarily, the above considerations show that for each pair $(\alpha, \beta) \in \mathbb{N} \times (-\mathbb{N})$ we may find points $p = k_1 a_1 + k_2 a_2 \in \curlywedge - \curlywedge$ and $p' = k_1' a_1 + k_2' a_2 \in \curlywedge - \curlywedge$ such that $\alpha = k_1 - k_1'$ and $\beta = k_2 - k_2'$. Due to rational independence of a_1 and a_2, for each $\varepsilon > 0$ there exist $p, p' \in \curlywedge - \curlywedge$ such that we have

$$p - p' \in (\curlywedge - \curlywedge - (\curlywedge - \curlywedge)) \cap B_\varepsilon(0).$$

Hence, $0 \in \mathrm{cl}\,((\curlywedge - \curlywedge - (\curlywedge - \curlywedge)) \setminus \{0\})$ which shows that $\curlywedge$ is not a Meyer set. $\square$

Chapter 10

Dynamical Properties

We want to establish a concept similar to the torus parametrization for dynamical Cut and Project Schemes. In the first section we provide simple conclusion regarding the dynamical hull of dynamical model sets, while in the second section we discuss the existence of a factor map.

10.1 Delone Dynamical Systems for Dynamical CPS

Applying the results of Chapter 9, in this short section we restate some of the lemmas of Section 4.3 in the setting of Dynamical Cut and Project Schemes.

An immediate consequence of Proposition 8.6 and Lemma 4.24 is the following

Proposition 10.1. *Let (X, T, G, φ) be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Assume φ is an FLC cocycle. Then $\Omega\left(\curlywedge_{x_0}^{T}(W)\right)$ is compact.*

Assuming that φ is even an FLC cocycle satisfying (APwCo), Proposition 4.26 and Lemma 8.8 yield

Proposition 10.2. *Let (X, T, G, φ) be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Assume φ is an FLC cocycle satisfying (APwCo). Then $\left(\Omega\left(\curlywedge_{x_0}^{T}(W)\right), G\right)$ is free.*

The following statement characterizes minimality and unique ergodicity of the dynamical hull.

Proposition 10.3. *Let (X, T, G, φ) be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$.*

(i) Assume that φ is an FLC cocycle. If x_0 is generic with respect to φ as well as generic with respect to W, then $\left(\Omega\left(\curlywedge_{x_0}^{T}(W)\right), G\right)$ is minimal.

(ii) Assume that φ is a UPF cocycle. Further, suppose T is a lattice in G, the space X is metrizable, (X, T, μ) is uniquely ergodic and each element of the corresponding partitions $\mathcal{P}(s)$ of X is regular. If W is regular, then $\left(\Omega\left(\curlywedge_{x_0}^{T}(W)\right), G\right)$ is uniquely ergodic.

Proof. Both assertions are a direct consequence of Proposition 9.2, Proposition 9.6 as well as Proposition 4.25. ☐

Finally, in the case of non-FLC sets, we obtain a statement regarding minimality of the hull.

Proposition 10.4. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with proper window $W \subseteq X$ and starting point $x_0 \in X$. Assume φ is a non-FLC cocycle. If x_0 is generic with respect to W as well as generic with respect to φ, then $\left(\Omega\left(\curlywedge_{x_0}^{T}(W)\right), G\right)$ is minimal.*

Proof. This is a direct consequence of Proposition 9.3 and Lemma 4.45. ☐

10.2 The Torus Parametrization for Dynamical CPS

In the following, we want to investigate the existence of a factor for dynamical hulls arising from dynamical Cut and Project Schemes similarly to the torus parametrization in case of classical Cut and Project Schemes.

To that end, consider a dynamical CPS $\mathcal{D} = (X, T, G, \varphi)$ with proper window $W \subseteq X$ and starting point $x_0 \in X$. Throughout this section we will assume that T is a lattice in G. Further, we assume that φ is an aperiodic FLC cocycle.This assumption guarantees by Proposition 10.1 and Proposition 10.2 that the hull $\left(\Omega\left(\curlywedge_{x_0}^{T}(W)\right), G\right)$ is a compact and free dynamical system.

It is not immediately clear which dynamical system might be a factor of $\left(\Omega\left(\curlywedge_{x_0}^{T}(W)\right), G\right)$. To investigate this problem, we will need a few additional assumptions and introduce some new concepts.

In the following we will discuss the concept of *suspensions*. Fix some $t \in T$ and define the mapping

$$h_t : G \times X \to G \times X : (g, x) \mapsto (g - \varphi_t(x), t \cdot x).$$

Due to continuity of the T-action on X as well as continuity of the group operation on G, the map h_t is continuous. Observe that h_t is bijective. Indeed, we have $h_t \circ h_t^{-1} = h_t^{-1} \circ h_t = \mathrm{id}$, where

$$h_t^{-1}(g, x) = (g + \varphi_t(t^{-1} \cdot x), t^{-1} \cdot x).$$

Note also that $h_{st} = h_s \circ h_t$. Thus, we have proven that

Lemma 10.5. *For each $t \in T$ the map h_t is a homeomorphism. Further, h defines a continuous T-action on $G \times X$.*

Now fix $c \in G$ and consider the mapping given by

$$(c, (g, x)) \mapsto (c + g, x).$$

Put

$$S_\varphi(X) = (G \times X)/\{h_t \mid t \in T\}$$

and let $\pi : G \times X \to S_\varphi(X)$ denote the canonical projection. Then it is easy to see that

$$(c, \pi(g, x)) \mapsto \pi(c + g, x)$$

is a continuous G-action on $S_\varphi(X)$. We call the dynamical system $(S_\varphi(X), G)$ the *suspension* of (X, T). By $[.,.]_{\sim,\varphi}$ we will denote the elements of $S_\varphi(X)$. If no confusion arises we will write $[.,.]_\sim$ instead of $[.,.]_{\sim,\varphi}$.

A simple calculation yields the following identities.

Lemma 10.6. *For all $t \in T$, $x \in X$ and $g, h \in G$ we have*

(i) $[g, x]_\sim = [h, t \cdot x]_\sim$ *if and only if* $g - h = \varphi_t(x)$.

(ii) $[0, t \cdot x]_\sim = [\varphi_t(x), x]_\sim$.

In general, the suspension might have some undesirable properties: [KMMS98] provides an example of a suspension which is not even Hausdorff. However, under our assumptions the space is rather well-behaved.

Lemma 10.7 ([FKMS93])**.** *The following holds:*

(i) $S_\varphi(X)$ *is a Hausdorff space.*

(ii) $S_\varphi(X)$ *is compact.*

(iii) $S_\varphi(X)$ *is a metric space.*

Proof. (i). Assume that $[g,x]_\sim$ and $[h,y]_\sim$ cannot be separated by open sets. We aim to show that we then obtain $[g,x]_\sim = [h,y]_\sim$. There exist sequences $(g_n,x_n)_{n\in\mathbb{N}} \subseteq G\times X$ and $(t_n)_{n\in\mathbb{N}} \subseteq T$ such that $\lim_{n\to\infty}(g_n,x_n) = (g,x)$ and $\lim_{n\to\infty}(g_n - \varphi_{t_n}(x_n), t_n\cdot x_n) = (h,y)$. Thus, $\lim_{n\to\infty}\varphi_{t_n}(x_n) = g-h$. By (Dist), we may assume that $t_n = t$ for all $n\in\mathbb{N}$. Thus, $(h,y) = (g-\varphi_t(x), t\cdot x)$ and hence $[g,x]_\sim = [h,y]_\sim$.

(ii). Since φ is an FLC cocycle we might choose a constant $C>0$ such that we have $\varphi_T(x)\cap B_C(g)\neq\emptyset$ for all $g\in G$ and $x\in X$. In particular, this yields the existence of a compact set $K = \mathrm{cl}(B_C(0))$ such that we have $\varphi_T(x)+K = G$ for all $x\in X$. Together with Lemma 10.6(ii) this yields

$$\{g\in G \mid g\cdot[0,x]_\sim \in \pi(B_\varepsilon(0)\times X)\} + K = G$$

for all $\varepsilon>0$. The claim follows by $S_\varphi(X) = K\cdot\pi(\mathrm{cl}(B_\varepsilon(0))\times X)$.

(iii). Observe that local compactness of G and compactness of X yield local compactness of the product space $G\times X$. Since the canonical projection is open, the suspension $S_\varphi(X)$ is also locally compact. By similar arguments we might show that $S_\varphi(X)$ is second-countable. On the other hand, since locally compact Hausdorff spaces are regular, the suspension is also a regular space. Now the claim follows by Urysohns Theorem. ☐

Remark 10.8. A metric on $S_\varphi(X)$ is given via the quotient metric.

A direct calculation yields

Lemma 10.9 ([FKMS93])**.** *The following holds:*

(i) The system $(S_\varphi(X),G)$ *is free.*

(ii) If (X,T) *is minimal, then* $(S_\varphi(X),G)$ *is minimal.*

Next, we need that W satisfies an additional property. To that end, put $\mathcal{I} = \{I\subseteq T \mid I \text{ is countable}\}$. We say W is *irredundant* if

$$\sharp\left(\bigcap_{s\in I} s^{-1}\cdot W \cap \bigcap_{s\in T\setminus I} s^{-1}\cdot \mathrm{cl}(X\setminus W)\right) \leq 1 \tag{10.2.1}$$

for all $I\in\mathcal{I}$, i.e., this intersection is either empty or contains one element.

In the following, we want to discuss the existence of irredundant windows. While it is not clear whether such windows exist in arbitrary dynamical systems, we may provide some information in case of equicontinuous systems (X,T). Recall that such systems carry the structure of a topological group by Lemma 3.5. Let $W\subseteq X$ be a proper subset such that $W+h=W$ implies $h=0$. Put $V=\mathrm{cl}(X\setminus W)$ and note that this set is clearly proper. Observe also that we have $\mathrm{int}(W\cap V)=\emptyset$. A pair (W,V) satisfying the three conditions mentioned before is also called a *separating cover* (compare [MP79] and [Pau76]).

In [MP79] it was shown that, in equicontinuous systems, the condition $W+h=W\Rightarrow h=0$ yields that for all distinct $x,y\in X$ there exists a time $t\in T$ such that $t\cdot x\in\mathrm{int}(W)$ and $t\cdot y\in\mathrm{int}(V)$. Now let $I\subseteq T$ be countable and assume there exist distinct

$$x,y\in\bigcap_{s\in I} s^{-1}\cdot W\cap\bigcap_{s\in T\setminus I} s^{-1}\cdot V. \tag{10.2.2}$$

Since (W,V) is a separating cover, we may find some $t\in T$ such that $t\cdot x\in\mathrm{int}(W)$ and $t\cdot y\in\mathrm{int}(V)$ which contradicts (10.2.2). Hence, we have shown the following.

Lemma 10.10. *Suppose* (X,T) *is a minimal and equicontinuous dynamical system and let* $W\subseteq X$ *be proper. If* $W+h=W$ *implies* $h=0$*, then* W *is irredundant in the sense of (10.2.1).*

In particular, for equicontinuous systems, irredundant windows in the sense of classical CPS provide a huge class of examples for irredundant windows in the sense of (10.2.1).

Before we construct a semiconjugation, we want to investigate the dynamical hull a bit further. To that end, assume we have given a Delone set Λ such that

$$\text{⅄}^T_{x_0}(\mathrm{int}(W)) \subseteq \Lambda \subseteq \text{⅄}^T_{x_0}(W).$$

Its dynamical hull may be parameterized by $G \times X$, i.e.,

Proposition 10.11. *Let $\mathcal{D} = (X, T, G, \varphi)$ be a DCPS with starting point $x_0 \in X$ and proper irredundant window $W \subseteq X$. Let $\Lambda \subseteq G$ be such that $\text{⅄}^T_{x_0}(\mathrm{int}(W)) \subseteq \Lambda \subseteq \text{⅄}^T_{x_0}(W)$. Then for any $\Gamma \in \Omega(\Lambda)$ there exists $(g, x) \in G \times X$ such that we have*

$$\text{⅄}^T_x(\mathrm{int}(W)) - g \subseteq \Gamma \subseteq \text{⅄}^T_x(W) - g.$$

Proof. Throughout the proof we assume that $\Omega(\Lambda) \subseteq \mathcal{U}_r$. Let Λ be as assumed. By Proposition 10.1 the hull $\Omega(\Lambda)$ is compact and hence for each $\Gamma \in \Omega(\Lambda)$ we may find a sequence $(g'_n)_{n\in\mathbb{N}} \subseteq G$ such that

$$\Gamma = \lim_{n\to\infty} \Lambda - g'_n.$$

Observe that for each g'_n we may choose an index $t_n \in T$ and some $g_n \in G$ such that we decompose $g'_n = g_n + \varphi_{t_n}(x_0)$. Put $\Lambda_n = \Lambda - g'_n$.

By the definition of the local topology we may find some $g \in G$ and compactum $K \subseteq G$ such that $\Gamma + g \cap K = \Lambda \cap K \neq \emptyset$. So in the following, instead of working with Γ, we may consider the element $\Gamma + g \in \Omega(\Lambda)$ such that $\Gamma + g = \lim_{n\to\infty} \Lambda - \varphi_{t_n}(x_0) - g_n$. Due to aperiodicity we may infer that $\lim_{n\to\infty} g_n = -g$. To simplify notation, without loss of generality we assume $g = 0$.

Therefore suppose that $\Lambda_n = \Lambda - \varphi_{t_n}(x_0)$ and put $x_n = t_n \cdot x_0$. Due to $\Lambda \subseteq \text{⅄}^T_{x_0}(W)$ we have

$$\Lambda_n = \Lambda - \varphi_{t_n}(x_0) \subseteq \text{⅄}^T_{x_0}(W) - \varphi_{t_n}(x_0) = \text{⅄}^T_{x_n}(W).$$

Thus, each point contained in Λ_n might be represented as $\varphi_s(x_n)$ for some $s \in T$. Fix $\varepsilon > 0$. Then there exists some $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$ we have $d_{\mathrm{LT}}(\Lambda_n, \Gamma) < \varepsilon$, i.e.,

$$\Lambda_n \cap B_{1/\varepsilon}(0) = \Gamma \cap B_{1/\varepsilon}(0). \tag{10.2.3}$$

By uniform discreteness and compactness of $B_{1/\varepsilon}(0)$, both intersections consist of finitely many points. Put

$$\Gamma \cap B_{1/\varepsilon}(0) = \{\gamma_1, \dots, \gamma_K\}.$$

By (10.2.3), for each γ_i there exists an index $s_i \in T$ such that we have $\gamma_i = \varphi_{s_i}(x_n)$. Similarly, we have $\gamma_i = \varphi_{t_i}(x_m)$ for $m > n$ and $t_i \in T$. Thus we have $\gamma_i = \varphi_{s_i}(x_n) = \varphi_{t_i}(x_m)$ and due to (APwCo) we acquire $s_i = t_i$.

Put $I_{n,\varepsilon} = \{s \in T \mid \varphi_s(x_n) \in \Lambda_n \cap B_{1/\varepsilon}(0)\}$. By (Dist), there exists some $\rho = \rho(\varepsilon) > 0$ such that $I_{n,\varepsilon} \subseteq B_\rho(0)$. Observe that we have $s \cdot x_n \in W$ for all $s \in I_{n,\varepsilon}$ and, on the other hand, $s \cdot x_n \notin \mathrm{int}(W)$ (and in particular $s \cdot x_n \in \mathrm{cl}(X \setminus W)$) for all $s \in B_\rho(0) \setminus I_{n,\varepsilon}$. Note that there may exist $s \in B_\rho(0) \setminus I_{n,\varepsilon}$ such that $s \cdot x_n \in \partial W = \partial(X \setminus W)$, however, such points are not included in $I_{n,\varepsilon}$ by definition.

By our previous considerations we have $I_\varepsilon := I_{n,\varepsilon} = I_{m,\varepsilon}$ for all $n, m > N_\varepsilon$. Further, for all $n, m > N_\varepsilon$ we have

$$x_n, x_m \in \bigcap_{s\in I_\varepsilon} s^{-1} \cdot W \cap \bigcap_{s \in B_\rho(0)\setminus I_\varepsilon} s^{-1} \cdot \mathrm{cl}(X \setminus W).$$

Note that $\lim_{\varepsilon\to 0} \rho(\varepsilon) = \infty$. By irredundancy of W, the diameter of the above intersection tends to zero as ε tends to zero. Together with completeness of X this yields the existence of a limit $x = \lim_{n\to\infty} x_n$ such that $s \cdot x \in W$ for all $s \in I_{n,\varepsilon}$ and $s \cdot x \in \mathrm{cl}(X \setminus W)$ for all $s \in B_\rho(0) \setminus I_{n,\varepsilon}$, $\varepsilon > 0$ and $n > N_\varepsilon$. Hence, $\Gamma = \lim_{n\to\infty} \Lambda_n \subseteq \lim_{n\to\infty} \text{⅄}^T_{x_n}(W) = \text{⅄}^T_x(W)$. Now assume that $p \in \text{⅄}^T_x(\mathrm{int}(W))$. Then for large $n \in \mathbb{N}$ we have $p \in \text{⅄}^T_{x_n}(\mathrm{int}(W)) \subseteq \Lambda_n$ and hence $p \in \Gamma$. $\square$

In the remaining part of this section we are going to construct the desired semiconjugation. To that end, let Λ be a Delone set such that

$$\curlywedge_{x_0}^{T}(\operatorname{int}(W)) \subseteq \Lambda \subseteq \curlywedge_{x_0}^{T}(W).$$

and fix $\Gamma \in \Omega(\Lambda)$. We define

$$\beta(\Gamma) = [g,x]_\sim :\Longleftrightarrow \curlywedge_{x}^{T}(\operatorname{int}(W)) - g \subseteq \Gamma \subseteq \curlywedge_{x}^{T}(W) - g. \tag{10.2.4}$$

As an immediate consequence of Proposition 10.11 we obtain

Corollary 10.12. *Let $\mathcal{D} = (X,T,G,\varphi)$ be a DCPS with starting point $x_0 \in X$ and proper window $W \subseteq X$. Assume that W is irredundant. Let $\Lambda \in \Omega$ such that $\curlywedge_{x_0}^{T}(\operatorname{int}(W)) \subseteq \Lambda \subseteq \curlywedge_{x_0}^{T}(W)$. Then the map $\beta : \Omega(\Lambda) \to S_\varphi(X)$ is well-defined.*

Now we aim to show that β is indeed a semiconjugation. Note that in the following proof we will make use of methods established in [Rob07, Theorem 5.19].

Lemma 10.13. *Under the assumptions of Corollary 10.12, the map $\beta : \Omega(\Lambda) \to S_\varphi(X)$ is uniformly continuous.*

Proof. First, we suppose that $\curlywedge_{x_0}^{T}(W)$ is an r-R-Delone set. Hence, all sets contained in $\Omega(\Lambda)$ are also r-R-Delone sets. Choose an arbitrary $\Gamma \in \Omega(\Lambda)$. Then there exist $g \in G$ and $x \in X$ such that $\Gamma \subseteq \curlywedge_{x}^{T}(W) - g$. We assume without loss of generality that $g = 0$ and put $I(\Gamma) = \{s \in T \mid \varphi_s(x) \in \Gamma\}$. By irredundancy of W we then have

$$\bigcap_{s\in I(\Gamma)} s^{-1}\cdot W \cap \bigcap_{s\in T\setminus I(\Gamma)} s^{-1}\cdot \operatorname{cl}(X\setminus W) = \{x\}.$$

For given $m > 0$ put

$$I_m = \bigcap_{s\in I(\Gamma\cap B_m(0))} s^{-1}\cdot W \cap \bigcap_{s\in B_\rho(0)\setminus I(\Gamma\cap B_m(0))} s^{-1}\cdot \operatorname{cl}(X\setminus W), \tag{10.2.5}$$

where $\rho = \rho(m) > 0$ is chosen according to (Dist). Clearly, we have $\lim_{m\to\infty} \operatorname{diam}(I_m) = 0$.

Now let $\Lambda_1, \Lambda_2 \in \Omega(\Lambda)$. Assume without loss of generality that $\Lambda_1 = \curlywedge_{x}^{T}(W) - g$ and $\Lambda_2 = \curlywedge_{y}^{T}(W) - h$, respectively (otherwise use that $\Lambda_1 \subseteq \curlywedge_{x}^{T}(W) - g$ and $\Lambda_2 \subseteq \curlywedge_{y}^{T}(W) - h$ in the following). Let $\varepsilon > 0$ and choose $m \in \mathbb{N}$ so large that $\operatorname{diam}(I_m) < \frac{\varepsilon}{2}$. Put

$$\delta = \min\left\{\frac{\varepsilon}{2}, \frac{1}{m+R}\right\}.$$

We suppose that $d_{\mathrm{LT}}(\Lambda_1,\Lambda_2) < \delta$. This means, there exists some $\kappa \in B_\delta(0)$ such that we have

$$\Lambda_1 \cap B_{1/\delta}(0) = (\Lambda_2 - \kappa) \cap B_{1/\delta}(0). \tag{10.2.6}$$

Since we assumed $\frac{1}{\delta} > R$, we have

$$\Lambda_1 \cap (\Lambda_2 - \kappa) \cap B_R(0) \neq \emptyset.$$

Thus, this intersection contains an element α. First, we have $\alpha \in \Lambda_1 = \curlywedge_{x}^{T}(W) - g$. Hence, there exists $p \in T$ such that $\alpha + g = \varphi_p(x)$. On the other hand, we also have $\alpha \in \Lambda_2 - \kappa$, which yields the existence of an index $q \in T$ such that $\alpha + h + \kappa = \varphi_q(y) \in \curlywedge_{y}^{T}(W)$. Note that we have $\Lambda_1 - \alpha = \curlywedge_{x}^{T}(W) - \varphi_p(x) = \curlywedge_{p\cdot x}^{T}(W)$ as well as $\Lambda_2 - \alpha - \kappa = \curlywedge_{y}^{T}(W) - \varphi_q(y) = \curlywedge_{q\cdot y}^{T}(W)$.

Then (10.2.6) together with $\frac{1}{\delta} > m + R$ yields

$$\curlywedge_{p\cdot x}^{T}(W) \cap B_m(0) = \curlywedge_{q\cdot y}^{T}(W) \cap B_m(0).$$

Write

$$I(\curlywedge^{T}_{p\cdot x}(W) \cap B_m(0)) = \{\varphi_{s_1}(p \cdot x), \dots, \varphi_{s_N}(p \cdot x)\}$$

and

$$I(\curlywedge^{T}_{q\cdot y}(W) \cap B_m(0)) = \{\varphi_{t_1}(q \cdot y), \dots, \varphi_{t_N}(q \cdot y)\},$$

respectively. Recall that by (Dist) there exists some $\rho = \rho(m) > 0$ such that both sets introduced above are contained in $B_\rho(0) \subseteq T$. By (APwCo) we have $s_i = t_i$ and, with the notation introduced in (10.2.5),

$$d_X(p \cdot x, q \cdot y) < \frac{\varepsilon}{2}.$$

By definition of β and Lemma 10.6 we observe that

$$\beta(\Lambda_1) = \beta(\curlywedge^{T}_{x}(W) - g) = [g, x]_\sim = [\alpha, p \cdot x]_\sim$$

as well as

$$\beta(\Lambda_2) = \beta(\curlywedge^{T}_{y}(W) - h) = [h, y]_\sim = [\kappa + \alpha, q \cdot y]_\sim.$$

Recall that $\kappa \in B_\delta(0)$. Using Remark 10.8 we compute

$$d_{S_\varphi(X)}(\beta(\Lambda_1), \beta(\Lambda_2)) \le \inf_{\substack{\phi \in [\beta(\Lambda_1)]_\sim \\ \psi \in [\beta(\Lambda_2)]_\sim}} d_{G\times X}(\phi, \psi) \le d_{G\times X}((\alpha, p \cdot x), (\kappa + \alpha, q \cdot y)) < \delta + \frac{\varepsilon}{2} < \varepsilon.$$

Hence, β is uniformly continuous.

□

Proposition 10.14. *Under the assumptions of Corollary 10.12, the map β is a semiconjugation.*

Proof. Let $c \in G$ and $\Gamma \in \Omega(\Lambda)$ such that $\beta(\Gamma) = [g, x]_\sim$. We may write $\Gamma = \lim_{n\to\infty} \Lambda - \varphi_{t_n}(x_0) - g_n$. By construction of β and Lemma 10.13 we then obtain

$$\begin{aligned}\beta(\Gamma - c) &= \beta\left(\lim_{n\to\infty} \Lambda - \varphi_{t_n}(x_0) - g_n - c\right) = \lim_{n\to\infty} \beta(\Lambda - \varphi_{t_n}(x_0) - g_n - c) \\ &= \lim_{n\to\infty} [c + g_n, t_n \cdot x_0]_\sim = c \cdot \lim_{n\to\infty} [g_n, t_n \cdot x_0]_\sim = c \cdot [g, x]_\sim.\end{aligned}$$

Furthermore, $\beta(\mathcal{O}(\Lambda)) = \mathcal{O}([0, x_0]_\sim)$. By Lemma 10.9, the system $(S_\varphi(X), G)$ is itself minimal. Since $\mathcal{O}([0, x_0]_\sim)$ is a dense subset of $S_\varphi(X)$, the function β is surjective and hence a semiconjugation. □

Chapter 11

Classical and Dynamical Euclidean CPS

In this chapter we will discuss the interplay of dynamical and classical Cut and Project Schemes in the Euclidean case. By Theorem 9.14, we may find dynamical CPS which yield Delone sets not satisfying the Meyer property. Thus, the class of dynamical model sets is in fact larger than the class of classical model sets.

Then again the question arises whether each classical model set has a representation as a dynamical model set. In Section 11.2 we will answer this question positively for the Euclidean case. For the convenience of the reader, we will give a rough outline of this approach for small dimensions in the following Section 11.1 (compare also the related discussions at the beginning of Sections 6.2 and 6.4).

11.1 Basic Observations

First, we consider the planar case $(\mathbb{R}, \mathbb{R}, \mathcal{L})$. We have already seen in Lemma 4.18 (compare also Section 6.2) that there exists an irrational matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$$

such that $\mathcal{L} = A(\mathbb{Z}^2)$. For the projection $\pi_2 : \mathcal{L} \to \mathbb{R}$ to the internal space, we then obtain (see also Section 6.4):

$$\begin{aligned} L^* = \pi_2(\mathcal{L}) &= \left\{ nc + md \mid (n, m) \in \mathbb{Z}^2 \right\} \\ &= \widetilde{\pi}^{-1}\left(\{ nc \mod d \mid n \in \mathbb{Z} \}\right) \\ &= d \cdot \pi^{-1}\left(\left\{ n \cdot \frac{c}{d} : n \in \mathbb{Z} \right\} \right), \end{aligned}$$

where $\widetilde{\pi} : \mathbb{R} \to \mathbb{R}/d\mathbb{Z}$ and $\pi : \mathbb{R} \to \mathbb{S}$ denote the canonical projections. By our assumptions on A we have $c/d \in \mathbb{R} \setminus \mathbb{Q}$. Hence, the projection of the irrational lattice to the internal space is the lift of an irrational rotation on $\mathbb{S}$ with rotation number c/d. We will denote this rotation by R_d.

Without loss of generality we will assume that d is positive, $c \in (0, d)$ as well as $W \subseteq [0, d]$. Additionally, we assume that a and $a - b$ have the same sign. In case this is not given, we might choose two other vectors $\begin{pmatrix} a' & b' \end{pmatrix}^T, \begin{pmatrix} c' & d' \end{pmatrix}^T \in \mathbb{R}^2$ which generate $\mathcal{L}$ and satisfy $\mathrm{sgn}(a') = \mathrm{sgn}(a' - b')$ as well as $d' > c'$. By rescaling, we may assume that $d = 1$ and write $R = R_1$. Put $X = \mathbb{S} = \mathbb{R}/\mathbb{Z}$. In an abuse of notation we will identify c with $\pi(c)$ throughout the following discussion.

First, we want to discuss the interplay between points in L^* and L. To that end, put $\mathcal{F} = [0, 1)$ and observe that $⅄(W) = \{p \in ⅄(\mathcal{F}) \mid p^* \in W\}$. Thus, the Delone set $⅄(\mathcal{F})$ contains all possible candidates of points of $⅄(W)$.

The preceding discussion yields the following: each $x \in \curlywedge(\mathcal{F})$ corresponds to a point $x^* \in \mathcal{F} \cap \mathcal{O}_{\mathbb{Z}}(0) \subseteq [0,1)$. More precisely, we have

$$nc \mod 1 \in \mathcal{F} \Longleftrightarrow nc - \lfloor nc \rfloor \in \mathcal{F} \Longleftrightarrow na - \lfloor nc \rfloor b \in \curlywedge(\mathcal{F})$$

(compare also the discussion leading to Equation (6.4.1)). In other words: iterating the starting point one time by c leads to increasing the corresponding point in the physical space one time by the value a, unless the iteration "exceeds" $\mathbb{S}$, in which case the corresponding point in the physical space is additionally decreased by b.

Again in other words: if we embed $\mathbb{S}$ canonically into $\mathcal{F} \subseteq \mathbb{R}$, for $x \in \mathcal{F} \cap L^*$ there are exactly two possible behaviors under translation of c:

(i) $x + c \in \mathcal{F}$ or

(ii) $x + c \in \mathcal{F} + 1$.

In the first case, we increase the corresponding point in $\curlywedge(\mathcal{F})$ by a, otherwise the corresponding point in the physical space is additionally decreased by b. Going back to $\mathbb{S}$, these cases translate to (by identifying x and $\pi(x)$)

(i') $x \in [0, 1-c) \subseteq \mathbb{S}$ or

(ii') $x \in [1-c, 1) \subseteq \mathbb{S}$, respectively.

Thus, if $x \in [0, 1-c)$, then one iteration by R does not "exceed" $\mathbb{S}$. Indeed, we have $R(x) \in [c, 1)$ and the corresponding point in $\curlywedge(X)$ is increased by a. In the second case, the iteration by R "exceeds" $\mathbb{S}$ and the corresponding point is additionally decreased by b.

Hence, we define subsets $P_1 = [0, 1-c)$ and $P_2 = [1-c, 1)$ of $\mathbb{S}$. Clearly, $\mathcal{P} = \{P_1, P_2\}$ is a partition of X. As seen in the examples discussed in Section 8.1,

$$\varphi : \mathbb{Z} \times X \to \mathbb{R} : (n, x) \mapsto \begin{cases} \sum_{k=0}^{n-1} a \cdot \chi_{P_1}(R^k(x)) + (a-b) \cdot \chi_{P_2}(R^k(x)) \text{ for } n > 0 \\ \sum_{k=n}^{-1} (-a) \cdot \chi_{P_1}(R^k(x)) + (b-a) \cdot \chi_{P_2}(R^k(x)) \text{ otherwise} \end{cases}$$

is a cocycle. As a consequence of the previous discussion we obtain

Proposition 11.1. *Given a planar CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ with window $W \subseteq \mathbb{R}$, there exists a dynamical CPS $(\mathbb{S}, \mathbb{Z}, \mathbb{R}, \varphi)$ such that*

$$\curlywedge(W) = \curlywedge_0^{\mathbb{Z}}(\pi(W)),$$

where $\pi : \mathbb{R} \to \mathbb{S}$ denotes the canonical projection.

Moreover, the dynamical system $(\mathbb{S}, R)$ introduced above together with the partition $\mathcal{P}$ is a Sturmian rotation. In particular, P_1 and P_2 are both bounded remainder sets. Thus, the set $\curlywedge_0^{\mathbb{Z}}(\pi(W))$ satisfies the Meyer property due to Theorem 9.14. This result is coherent with Lemma 4.8.

11.2 Connection between Dynamical and Euclidean CPS

Consider an Euclidean Cut and Project Scheme $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$ with compact window $W \subseteq \mathbb{R}^M$. Similarly to the low-dimensional case, we want to use the lattice $\mathcal{L}$ to construct a dynamical system and a corresponding cocycle which describe all possible candidates of points contained in $\curlywedge(W)$. It turns out that this dynamical system is in fact conjugated to a minimal $\mathbb{Z}^N$-action (which is given as an N-fold irrational rotation) on the torus $\mathbb{T}^M$.

First, we have to ensure that we may find primitive vectors of $\mathcal{L}$ such that the convex hull of their projection to $\mathbb{R}^M$ fully contains the window W. These projected vectors then give rise to a lattice in $\mathbb{R}^M$ which yields in turn the desired dynamical system.

To that end we will assume without loss of generality that the window does not contain the origin, that is, we suppose that $0 \notin \operatorname{int}(\operatorname{Conv}(W))$ holds. Otherwise, by denseness of L^*, we may find some appropriate $h^* \in L^*$ such that $W' = W + h^*$ satisfies the above condition. In such a case, we obtain $人(W)$ via $人(W) = 人(W') - h$.

Lemma 11.2. *Let $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$ be a CPS with compact window $W \subseteq \mathbb{R}^M$. Then there exist M primitive vectors $\delta_1, \ldots, \delta_M \in \mathbb{R}^{N+M}$ such that $W \subseteq \operatorname{Conv}\{0, \pi_{\mathbb{R}^M}(\delta_1), \ldots, \pi_{\mathbb{R}^M}(\delta_M)\}$.*

Proof. In the following, we denote the canonical projection of a vector in $\mathbb{R}^{N+M}$ to its i-th coordinate by $\pi_i : \mathbb{R}^{N+M} \to \mathbb{R}$. Lemma 4.15 and Remark 4.16 yield the existence of a primitive vector $\delta_1 \in \mathbb{R}^{N+M}$ satisfying

$$|\pi_{N+1}(\delta_1)| > \max_{w \in \operatorname{Conv}(W)} |\pi_{N+1}(w)|. \tag{*}$$

Fix δ_1 and consider the set of all remaining primitive vectors $\mathcal{S} = \{\widetilde{\delta}_2, \ldots, \widetilde{\delta}_{N+M}\}$ of $\mathcal{L}$. If there is a vector $\delta \in \mathcal{S}$ satisfying $|\pi_{N+2}(\delta)| > \max_{w \in \operatorname{Conv}(W)} |\pi_{N+2}(w)|$, put $\delta_2 = \delta$. Then proceed successively to find vectors in $\widetilde{\delta} \in \mathcal{S} \setminus \{\delta\}$ satisfying

$$|\pi_{N+j}(\widetilde{\delta})| > \max_{w \in \operatorname{Conv}(W)} |\pi_{N+j}(w)| \tag{$\sharp$}$$

for $j = 3, \ldots, M$.

However, if at one point there is no primitive vector satisfying ($\sharp$), we choose a new primitive vector $\delta_1' \in \mathbb{R}^{N+M} \setminus (\{\delta_1\} \cup \mathcal{S})$ satisfying $(*)$ and proceed as described before. Since the lattice admits arbitrarily many primitive vectors with growing length, this algorithm has to determine eventually. ☐

Now fix M vectors $\delta_1, \ldots, \delta_M$ according to the previous lemma. Note that we may assume that the projections $\omega_i = \pi_{\mathbb{R}^M}(\delta_i)$ of those vectors are linearly independent (compare also Remark 4.17).

Let $v_1, \ldots, v_N$ denote the remaining primitive vectors of $\mathcal{L}$. In particular, we have that

$$\mathcal{L} = \sum_{i=1}^{N} \mathbb{Z} v_i + \sum_{i=1}^{M} \mathbb{Z} \delta_i.$$

Now put

$$B = \begin{pmatrix} \omega_1 & \ldots & \omega_M \end{pmatrix} \in \mathrm{GL}(M, \mathbb{R}).$$

Then B generates a lattice Γ in the internal space, i.e., $\Gamma = B(\mathbb{Z}^M) \subseteq \mathbb{R}^M$. By our choice of B the window W is fully contained in the fundamental domain

$$\mathcal{F} = \left\{ \sum_{i=1}^{M} t_i \omega_i : t_i \in [0,1] \right\}$$

of Γ. Now we define a topological space

$$X = \mathbb{R}^M / \Gamma$$

and let $\pi : \mathbb{R}^M \to X$ denote the canonical quotient map. Note that B induces a homeomorphism between the M-dimensional standard torus $\mathbb{T}^M$ and X, i.e.,

$$\mathbb{T}^M \to X : [z]_{\mathbb{Z}^M} \mapsto [Bz]_X,$$

where $[.]_{\mathbb{Z}^M}$ and $[.]_X$ denote the corresponding equivalence classes on $\mathbb{T}^M$ and X, respectively. For $i = 1, \ldots, N$ we define

$$c_i = \pi(v_i).$$

Then we may define a $\mathbb{Z}^N$-action on X given by

$$R : \mathbb{Z}^N \times X \to X : (n_1, \ldots, n_N, x) \mapsto x + \sum_{i=1}^{N} n_i c_i \mod \Gamma. \tag{11.2.1}$$

Applying Lemma 4.18 yields minimality of this action. Furthermore, we define the *i-th subrotation* as

$$R_i : X \to X : x \mapsto x + c_i \mod \Gamma.$$

Observe that we have $R(x) = R_1 \circ \ldots \circ R_N(x)$ as well as $R_i \circ R_j = R_j \circ R_i$ for all $i, j \in \{1, \ldots, N\}$ (compare also the second example in Section 8.1).

In the following we aim to construct the desired cocycle. To that end, we first construct partitions of X which we will then use to define the cocycle.

Choose representatives $c_i' \in \pi^{-1}(c_i) \cap \mathcal{F}$ for $i = 1, \ldots, N$. Given a point $x \in \mathcal{F}$, there are 2^M possible sets x may gets transported to by translation with c_i', i.e., for each $i = 1, \ldots, N$ we have

$$x + c_i' \in \mathcal{F} + \sum_{k=1}^{M} \tau_k \omega_k \tag{11.2.2}$$

where $\tau_k \in \{0, 1\}$. For each $i = 1, \ldots, N$ we define 2^M subsets of X given by

$$P_l^i = \pi \left(\mathcal{F} \cap \left(\mathcal{F} + \sum_{k=1}^{M} \tau_k^l \omega_k \right) - c_i' \right)$$

where $\tau^l \in \{0, 1\}^M$ and τ_k^l denotes the k-th entry of τ^l. Put

$$\mathcal{P}^i = \left\{ P_l^i \mid l = 1, \ldots, 2^M \right\}. \tag{11.2.3}$$

It is not hard to see that the following lemma holds.

Lemma 11.3. *For each $i = 1, \ldots, N$, the set system $\mathcal{P}^i$ is a partition of X. In particular we have*

(i) $\bigcap_{l=1}^{2^M} P_l^i = \emptyset$,

(ii) $\bigcup_{l=1}^{2^M} P_l^i = X$.

In the following, put $a_i = \pi_{\mathbb{R}^N}(v_i)$ for $i = 1, \ldots, N$ as well as $a_{N+i} = \pi_{\mathbb{R}^N}(\delta_i)$ for $i = 1, \ldots, M$. Given $i \in \{1, \ldots, N\}$, to each $P_l^i \in \mathcal{P}^i$ we assign a *steplength*

$$s_l^i = a_i - \sum_{k=1}^{M} \tau_k^l a_{N+k}.$$

Further, put

$$f_i : X \to \mathbb{R}^N : x \mapsto \sum_{l=1}^{2^M} s_l^i \chi_{P_l^i}(x). \tag{11.2.4}$$

We obtain the following.

Lemma 11.4. *The N functions f_i given as in (11.2.4) induce a cocycle $\varphi : \mathbb{Z}^N \times X \to \mathbb{R}^N$. Moreover, the cocycle is given by*

$$\varphi_{(n_1, \ldots, n_N)}(x) = \sum_{i=1}^{N} \left(\sum_{j=0}^{n_i - 1} \left(\sum_{l=1}^{2^M} s_l^i \chi_{P_l^i} \left(R_i^j (R_{i+1}^{n_{i+1}} \circ \ldots \circ R_N^{n_N}(x)) \right) \right) \right). \tag{11.2.5}$$

Proof. Let $x \in X$ and $i, j \in \{1, \ldots, N\}$. We aim to show that

$$f_i(x) + f_j(R_i(x)) = f_i(R_j(x)) + f_j(x)$$

holds. Then the claim holds according to the discussions in Section 8.1. For given i we have

$$f_i(x) = a_i - \sum_{k=1}^{M} \left(a_{N+k} \sum_{l=1}^{2^M} \tau_k^l \chi_{P_l^i}(x) \right)$$

and thus we aim to show that

$$\begin{aligned} &a_i - \sum_{k=1}^{M} \left(a_{N+k} \sum_{l=1}^{2^M} \tau_k^l \chi_{P_l^i}(x) \right) - a_j + \sum_{k=1}^{M} \left(a_{N+k} \sum_{l=1}^{2^M} \tau_k^l \chi_{P_l^j}(x) \right) \\ =&a_i - \sum_{k=1}^{M} \left(a_{N+k} \sum_{l=1}^{2^M} \tau_k^l \chi_{P_l^i}(R_j(x)) \right) - a_j + \sum_{k=1}^{M} \left(a_{N+k} \sum_{l=1}^{2^M} \tau_k^l \chi_{P_l^j}(R_i(x)) \right). \end{aligned}$$

Recall that, by Lemma 4.18), $(a_i)_{i=1}^{N+M}$ is a family of rationally independent vectors. Thus, the above equality holds if for each $k \in \{1, \ldots, M\}$ we have that

$$\sum_{l=1}^{2^M} \tau_k^l (\chi_{P_l^j}(x) - \chi_{P_l^i}(x)) = \sum_{l=1}^{2^M} \tau_k^l (\chi_{P_l^j}(R_i(x)) - \chi_{P_l^i}(R_j(x))).$$

To see that this equality holds we consider the case of a $\mathbb{Z}^2$-action on $\mathbb{S}$ given by $(n_1, n_2, x) \mapsto x + n_1\alpha + n_2\beta \mod \mathbb{S}$ for two rational independent numbers $\alpha, \beta \in [0, 1)$. In this case, a simple case analysis shows that we always have

$$\chi_{[1-\beta,1)}(x) - \chi_{[1-\alpha,1)}(x) = \chi_{[1-\beta,1)}(x + \alpha \mod \mathbb{S}) - \chi_{[1-\alpha,1)}(x + \beta \mod \mathbb{S}).$$

As pointed out at the beginning of this section, the dynamical system $(X, \mathbb{Z}^N)$ is conjugated to $(\mathbb{T}^M, H)$, where H is an irrational rotation induced by N rationally independent rotation vectors $w_n = Bc_n$, $n \in \{1, \ldots, N\}$. Now we may write $w_i = \begin{pmatrix} \alpha_1^i & \ldots & \alpha_M^i \end{pmatrix}^T$, where the α_n^i are itself rationally independent. This yields that each P_l^i is homeomorphic to a product $\prod_{j=1}^{2^M} C_j$, where $C_j \in \{[0, 1 - \alpha_j^i), [1 - \alpha_j^i, 1)\}$. The same observation holds for the rotation along R_j and the corresponding partition $\mathcal{P}^j$. Applying the low-dimensional observation successively yields the desired equality. ☐

Altogether, the preceding constructions yield the following.

Theorem 11.5. *Given an Euclidean Cut and Project Scheme* $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$*, there exists a dynamical Cut and Project Scheme* $(X, \mathbb{Z}^N, \mathbb{R}^N, \varphi)$ *such that for any window* $W \subseteq \mathbb{R}^M$ *there exists a set* $V \subseteq X$ *such that we have*

$$\curlywedge(W) = \curlywedge_0^{\mathbb{Z}^N}(V).$$

Proof. As discussed in the beginning of this section, we assume without loss of generality that $0 \notin \operatorname{int}(\operatorname{Conv}(W))$. We assume that $\mathcal{L}$ can be represented as $\mathcal{L} = \sum_{i=1}^N \mathbb{Z}v_i + \sum_{i=1}^M \mathbb{Z}\delta_i$, where the δ_i are chosen according to Lemma 11.2. Hence, we may construct a lattice $\Gamma \subseteq \mathbb{R}^M$ and hence a topological space $X = \mathbb{R}^M/\Gamma$. Further, we define a rotation $R : X \to X$ as given in (11.2.1) and a cocycle $\varphi : \mathbb{Z}^N \times X \to X$ as given in (11.2.5). Recall that all those constructions depend on the choice of the irrational matrix A with $\mathcal{L} = A(\mathbb{Z}^{N+M})$, which in turn depends on the window $W \subseteq \mathbb{R}^M$.

In the following, put $V = \pi(W)$, where $\pi : \mathbb{R}^M \to X$ denotes the canonical projection. Now let $x \in \curlywedge_0^{\mathbb{Z}^N}(V)$. Then there exists some $\boldsymbol{n} = n(x) = (n_1, \ldots, n_N) \in \mathbb{Z}^N$ such that $\varphi_{\boldsymbol{n}}(0) = x$. By definition of φ (and the convention of writing sums with negative upper bounds introduced in Section 8.1) we have

$$x = \sum_{i=1}^{N} \left(\sum_{j=0}^{n_i - 1} \left(\sum_{l=1}^{2^M} s_l^i \chi_{P_l^i}(R_i^j(R_{i+1}^{n_{i+1}} \circ \ldots \circ R_N^{n_N}(0))) \right) \right).$$

Due to Lemma 11.3, for all $i = 1, \ldots, N$ there exist unique integers

$$\alpha_l^i = \sum_{j=0}^{n_i - 1} \chi_{P_l^i}(R_i^j(R_{i+1}^{n_{i+1}} \circ \ldots \circ R_N^{n_N}(0)) \in \mathbb{Z}$$

such that we have $\sum_{l=1}^{2^M} \alpha_l^i = n_i$ as well as

$$x = \sum_{i=1}^{N} \sum_{l=1}^{2^M} \alpha_l^i s_l^i.$$

Applying the definition of the steplengths, a straightforward calculation yields

$$x = \sum_{i=1}^{N} a_i \sum_{l=1}^{2^M} \alpha_l^i - \left(\sum_{k=1}^{M} a_{N+k} \sum_{i=1}^{N} \sum_{l=1}^{2^M} \alpha_l^i \tau_k^l \right).$$

By defining

$$\beta_k = \sum_{i=1}^{N} \sum_{l=1}^{2^M} \alpha_l^i \tau_k^l \in \mathbb{Z}$$

we obtain the identity

$$x = \sum_{i=1}^{N} a_i n_i - \sum_{k=1}^{M} a_{N+k} \beta_k. \tag{11.2.6}$$

Let $z = (n_1, \ldots, n_N, -\beta_1, \ldots, -\beta_M) \in \mathbb{Z}^{N+M}$. Then identity (11.2.6) implies $x = \pi_1(Az)$ and hence $x \in L = \pi_1(\mathcal{L})$.

Due to our assumption, we have $R^{\boldsymbol{n}}(0) = R_1^{n_1} \circ \ldots \circ R_N^{n_N}(0) \in V = \pi(W)$, that is,

$$\sum_{i=1}^{N} n_i c_i \mod \Gamma = \sum_{i=1}^{N} \left(c_i \sum_{l=1}^{2^M} \alpha_l^i \right) \mod \Gamma \in V.$$

Thus,

$$\pi^{-1} \left(\sum_{i=1}^{N} \left(c_i \sum_{l=1}^{2^M} \alpha_l^i \right) \right) \subseteq \pi^{-1}(V).$$

Consider the point

$$y = \sum_{i=1}^{N} c_i' \sum_{l=1}^{2^M} \alpha_l^i - \sum_{k=1}^{M} \beta_k \omega_k \in \pi^{-1}(R^{\boldsymbol{n}}(0)).$$

Recall that W was supposed to be contained in a fundamental domain $\mathcal{F}$ of Γ where $0 \in \mathcal{F}$. Lifting the action on the torus X to $\mathbb{R}^M$ means that we iterate 0 to the point $\sum_{i=1}^{N} n_i c_i' \in \mathcal{F} + \sum_{k=1}^{M} \beta_k \omega_k$. By definition, the numbers β_k describe the amount of translations we need to translate $\sum_{i=1}^{N} n_i c_i'$ back into $\mathcal{F}$. Indeed, we have

$$y = \pi_2(Az) \in W.$$

Hence, $x \in \{p \in \mathbb{R}^N \mid p^* \in W\} = \curlywedge(W)$.

For the converse inclusion, put $\kappa_i = \pi_{\mathbb{R}^M}(v_i)$ for $i = 1, \ldots, N$ and $\kappa_{N+i} = \omega_i$ for $i = 1, \ldots, M$. Now suppose $x \in \curlywedge(W)$. Then there exists a unique vector $(\alpha_1, \ldots, \alpha_{N+M}) \in \mathbb{Z}^{N+M}$ such that

$$x = \sum_{k=1}^{N+M} \alpha_k a_k \in L \text{ as well as } x^* = \sum_{k=1}^{N+M} \alpha_k \kappa_k \in W.$$

By definition of the lattice Γ we obtain that

$$\sum_{k=1}^{M} \alpha_k \kappa_{N+k} \mod \Gamma = \sum_{k=1}^{M} \alpha_k \omega_k \mod \Gamma = 0. \tag{11.2.7}$$

Now let $\boldsymbol{n} = n(x) = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N$. Then we have

$$\pi(x^*) = R^{\boldsymbol{n}}(0) \in \pi(W) = V.$$

Put

$$\widetilde{R}_i^j(0) = R_i^j(R_{i+1}^{\alpha_{i+1}} \circ \ldots \circ R_N^{\alpha_N}(0)).$$

We calculate

$$\begin{aligned}
\varphi_{\boldsymbol{n}}(0) &= \sum_{i=1}^{N} \left(\sum_{j=0}^{\alpha_i - 1} \left(\sum_{l=1}^{2^M} s_l^i \chi_{P_l^i}(\widetilde{R}_i^j(0)) \right) \right) \\
&= \sum_{i=1}^{N} \left(\sum_{j=0}^{\alpha_i - 1} \left(\sum_{l=1}^{2^M} \left(a_i - \sum_{m=1}^{M} \tau_m^l a_{N+m} \right) \chi_{P_l^i}(\widetilde{R}_i^j(0)) \right) \right) \\
&= \sum_{i=1}^{N} \left(\sum_{j=0}^{\alpha_i - 1} \left(a_i \sum_{l=1}^{2^M} \chi_{P_l^i}(\widetilde{R}_i^j(0)) - \sum_{l=1}^{2^M} \left(\sum_{m=1}^{M} \tau_m^l a_{N+m} \right) \chi_{P_l^i}(\widetilde{R}_i^j(0)) \right) \right).
\end{aligned}$$

To conclude the proof, we want to discuss both summands in the above expression separately.

Fix some $i \in \{1, \ldots, N\}$. Due to $\mathcal{P}^i$ being a partition of X consisting of 2^M elements, we have $\sum_{l=1}^{2^M} \chi_{P_l^i}(\widetilde{R}_i^j(0)) = 1$ for each $j \in \mathbb{Z}$. In particular this yields that

$$a_i \sum_{j=0}^{\alpha_i - 1} \sum_{l=1}^{2^M} \chi_{P_l^i}(\widetilde{R}_i^j(0)) = a_i \sum_{j=0}^{\alpha_i - 1} \sum_{l=1}^{2^M} \chi_{P_l^i}(R_i^j(R_{i+1}^{\alpha_{i+1}} \circ \ldots \circ R_N^{\alpha_N}(0))) = a_i \alpha_i$$

and thus

$$\varphi_{\boldsymbol{n}}(0) = \sum_{i=1}^{N} a_i \alpha_i - \sum_{i=1}^{N} \left(\sum_{j=0}^{\alpha_i - 1} \left(\sum_{l=1}^{2^M} \left(\sum_{m=1}^{M} \tau_m^l a_{N+m} \right) \chi_{P_l^i}(\widetilde{R}_i^j(0)) \right) \right). \tag{11.2.8}$$

Now consider the second expression

$$\sum_{j=0}^{\alpha_i - 1} \left(\sum_{l=1}^{2^M} \left(\sum_{m=1}^{M} \tau_m^l a_{N+m} \right) \chi_{P_l^i}(\widetilde{R}_i^j(0)) \right). \tag{11.2.9}$$

for some fixed $i \in \{1, \ldots, N\}$. Put $\xi = R_{i+1}^{\alpha_{i+1}} \circ \ldots \circ R_{\alpha_N}^{\alpha_N}(0) \in X$ and fix some representative ζ of $\pi^{-1}(\xi)$ such that $\zeta \in \mathcal{F}$. Recalling (11.2.2), any iteration of ξ by R_i corresponds to a translation of ζ by c_i'. Thus, by construction of $\mathcal{P}^i$, if ξ is contained in any partition element $\mathcal{P}_l^i$, where $l \neq 1$, iteration by R_i corresponds to ζ leaving $\mathcal{F}$ in direction of $\sum_{m=1}^{M} \tau_m^l \omega_m$.

Hence, according to Equation (11.2.7), we may rewrite the sum in (11.2.9) as

$$\sum_{m=1}^{M} a_{N+m} \sigma_m^i,$$

where the $\sigma_m^i \in \mathbb{Z}$ are integers depending on α_i. Summarized, the numbers σ_m^i represent how often we have to decrease $\zeta + \alpha_i \cdot c_i'$ by ω_m to be contained again in $\mathcal{F}$, i.e., we have $\zeta + \alpha_i \cdot c_i' - \sum_{m=1}^{M} \sigma_m^i \omega_m \in \mathcal{F}$.

By defining $\sigma_m = \sum_{i=1}^{N} \sigma_m^i$, we then may rewrite (11.2.8) as

$$\varphi_{\boldsymbol{n}}(0) = \sum_{i=1}^{N} a_i \alpha_i + \sum_{i=1}^{M} a_{N+i} \sigma_i.$$

However, due to our previous discussion and $x^* \in W \subseteq \mathcal{F}$, the values of the σ_i are exactly given by the α_{N+i}. This immediately yields

$$x = \sum_{i=1}^{N+M} a_i \alpha_i = \sum_{i=1}^{N} a_i \alpha_i + \sum_{i=1}^{M} a_{N+i} \sigma_i = \varphi_{\boldsymbol{n}}(0)$$

and hence $x \in \curlywedge_0^{\mathbb{Z}^N}(V)$, which concludes the proof. $\square$

Remark 11.6. Observe that the above proof also holds for Euclidean Cut and Project Schemes whose windows have empty interior. According to Remark 8.2, the corresponding dynamical model set is only a weak model set, although the cocycle itself would provide relatively dense point sets in case of windows with non-empty interior.

Part IV

Conclusion and Outlook

Chapter 12

Open Problems

Concluding this thesis, we want to collect some open problems which arose in the context of our investigations.

12.1 Open Problems in Part II

The most obvious question for all constructions done in Chapter 5 is the question for higher-dimensional internal spaces. While in our case of $H = \mathbb{R}$ the boundary of each irregular proper window has to be a Cantor set, in higher dimensions there is much greater variety of possible structures. This is due to the fact that higher-dimensional boundaries need to contain non-trivial connected components. Since the window used in Theorem 5.18 was constructed out of an arbitrary Cantor set with positive measure, it might be difficult to obtain general statements like this for higher-dimensional internal spaces.

Problem 1. *Is it possible to obtain statements similar to Theorem 5.18 for Euclidean CPS* $(\mathbb{R}^N, \mathbb{R}^M, \mathcal{L})$*?*

However, instead of giving general statements it should be possible to construct specific examples with only minor modifications of the one-dimensional case. For instance, in the probabilistic setting, we might start the construction of a irregular proper window with a Sierpinski carpet of positive measure. By labeling the squares which were removed in the construction of the carpet and including each of them into the window with probability $1/2$, we should obtain an example for an irregular proper window whose associated dynamical hull yields positive topological entropy almost surely.

In case of the deterministic constructions, which lead to Theorems 5.23 as well as 5.25, we could obtain similar statements in higher dimensions by slightly modifying the given methods. By starting the construction with the projection of a fundamental domain of the lattice $\mathcal{L}$ to $H = \mathbb{R}^M$, removing neighbourhoods of rapidly decreasing size around points in L^* should yield an initial Cantor set C_0. Then we just have to paste in locally topologically independent sets into these gaps to obtain (weak) model sets yielding hulls with positive entropy.

The construction of the boundary of the irregular window in Chapter 6 depended heavily on the irregular matrix $A = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$. In fact, the boundary was supposed to be a self similar Cantor set which was constructed out of an irrational rotation induced by c on $\mathbb{S}$. As seen for instance in the discussions in Chapter 11 or in Section 6.4, there is an interplay between the dimension of the physical space and an action on a fitting quotient of the internal space, i.e., a higher-dimensional physical space $\mathbb{R}^N$ yields a $\mathbb{Z}^N$-action on $\mathbb{S}$. In our case we might describe such an $\mathbb{Z}^N$-action by the sum of N irrational rotations on $\mathbb{S}$.

Thus, to obtain a result about vanishing entropy without using Proposition 6.25, we would need to construct a self similar Cantor set $C \subseteq \mathbb{S}$ out of N rationally independent rotations on $\mathbb{S}$.

Problem 2. *Is it possible to construct self similar windows for CPS* $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$*?*

Associated to the question above, another problem arises. Recalling the discussions in Section 6.4, we associated N planar CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L}_i)$ to a given higher-dimensional CPS $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$. Note that all occurring CPS had the same window $W \subseteq \mathbb{S}$. It turned out that the entropy of the system $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ vanishes as long as the entropy of at least one of the hulls arising from an associated planar CPS is zero. This leads to the following question:

Problem 3. *For a given CPS* $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$*, is it possible to construct an irregular window* $W \subseteq \mathbb{S}$*, such that one of the associated planar hulls has zero entropy while another planar hull has positive entropy?*

The last question is quite mandatory:

Problem 4. *Is it possible to generalize the constructions of Chapter 6 to higher-dimensional internal spaces?*

12.2 Open Problems in Part III

In the beginning of Part III we defined dynamical Cut and Project Schemes and discussed properties of the associated cocycle. A first general question is the following:

Problem 5. *Is it possible to drop assumptions on the cocycle for all results obtained in Part III?*

In case of induced cocycles, we discussed an example in Section 8.2. Even in higher dimensions, cocycles induced by step functions with certain coefficients are FLC cocycles. However, it is unclear which conditions have to hold for the coefficients of such functions such that the obtained cocycle satisfies certain properties, i.e.,

Problem 6. *Consider a step function* $f : X \to \mathbb{R}^N$*. Which conditions need its coefficients to satisfy such that the induced cocycle is a Delone cocycle, an FLC cocycle or a UPF cocycle?*

In Section 9.1, we provided a statement regarding repetitivity of non-FLC sets (compare Proposition 9.3). This was possible due to the notation of almost repetitivity of non-FLC sets and the proper connection between almost repetitivity and minimality of the corresponding hull. Clearly, we may ask for a similar statement regarding uniform patch frequencies and unique ergodicity.

Problem 7. *Is it possible to define an analogue to uniform patch frequencies for non-FLC sets such that the corresponding hull is uniquely ergodic in this case? Can we obtain a statement similar to Proposition 9.6 for non-FLC sets?*

More open problems arise from the discussion of bounded remainder sets and their relation to Meyer sets. First of all, we considered only $\mathbb{Z}^N$-rotations on a compact space X. It turned out that there is a characterization of bounded remainder sets on the M-dimensional torus. Thus, it seems obvious to ask whether such sets exist for non-equicontinuous dynamical systems.

Even more general, given a group rotation (X, T), the question arises if it is possible to define a group-theoretic analogue for bounded remainder sets. Clearly, this notation should somehow depend on an averaging sequence. This leads also to the question whether such a generalization can be independent of the choice of the averaging sequence or not. Again the question arises how such a generalization could be done in case (X, T) is not a group rotation.

Finally, it was crucial for our proofs that the cocycle φ was induced by step functions. We may ask how a proof similar to Theorem 9.14 works in a more abstract setting.

Problem 8. *Can we find examples for bounded remainder sets for arbitrary* $\mathbb{Z}^N$*-actions? Is it possible to generalize this notation to arbitrary group action* (X, T)*? In this case, is it possible to obtain a similar statement to Theorem 9.14?*

On the converse, we obtained examples for non-Meyer sets arising from DCPS. Even more, the partitions introduced in the discussions in Section 11.2 are not bounded remainder sets according to Proposition 9.13. However, the resulting sets satisfy the Meyer property. Hence, it might turns out to be fruitful to investigate the connection between the underlying dynamics and the Meyer property further. It might be of interest to find an answer to

Problem 9. *What are necessary conditions for a DCPS to yield Meyer sets?*

In Chapter 10 we discussed the existence of a factor of dynamical hulls. A crucial assumption was irredundancy of the window. As we have seen, there was a huge class of irredundant windows in the case of equicontinuous dynamical systems. However, it is still unclear whether such windows exist in arbitrary dynamical systems.

Problem 10. *Given an arbitrary dynamical system (X, T), under which assumptions on X and T exists an irredundant window $W \subseteq X$?*

Further investigation can also be done regarding the structure of the factor we obtained.

Problem 11. *Which dynamical properties does the factor space $(S_\varphi(X), G)$ have?*

In Chapter 11 we showed that model sets arising from Euclidean CPS have a representation as dynamical model sets. More general, we ask

Problem 12. *Let $(G, H, \mathcal{L})$ be a CPS with compact window $W \subseteq H$. Does there always exist a dynamical Cut and Project Scheme (X, T, G, φ), a compact subset $V \subseteq X$ and a point $x \in X$ such that we have $\curlywedge(W) = \curlywedge_x^T(V)$?*

Of course, if we consider groups not admitting a metric, we have to adjust the properties of the cocycle to such a setting.

We expect that a solution to this problem gives more insight into the structure of dynamical model sets. In particular, we might obtain a partial solution to Problem 9.

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Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts hat mich Prof. Dr. Tobias Oertel-Jäger unterstützt.

Jena, den

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